OPERATING MANUAL FOR SYNCHRO OPERATED DIFFERENTIAL ANALYSER

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I. INTRODUCTION

A synchro operated differential analyser, or mechanical analog computer for solving systems of ordinary differential equations, has been developed and built by the author, with much valuable assistance from J. Snyder, R. Norberg, and others, at the University of Illinois. The emphasis in the development was on convenience and simplicity of use and on small size, portability, and economy. It is believed that most scientific research workers with a reasonable knowledge of differential equations can use the computer effectively alone, after study of this manual, some practice at planning machine set-ups and a few hours of supervised operation.

Section II of this manual is a description of the computer with emphasis on the logical structure and on other points which must be understood in order to operate it successfully. Details of the mechanical design and other matters relating to care and maintenance are not covered. Section III gives a standard step by step routine for setting up and solving a problem, together with reasons why the operations should be performed in the given order. Anyone who plans to use the machine more than very occasionally should make it a habit to follow this routine in order to avoid lost motions and too many spoiled problem solutions. Section IV covers the subject of planning the computer set-up for solving any given problem, which is done on a paper blank available for this purpose. The integrator factor, the mechanical limits of motion of the integrators and plotting tables and the subject of scale factors, all of which are important in planning the set-up, are treated in this section. Section V contains examples of set-ups for a variety of specific problems.

The reader will find that a mere passive knowledge of the material in this manual is not sufficient to enable him to operate the machine effectively. This is especially the case with planning set-ups with workable scale factors. The reader is advised to plan set-ups for the problems in Section V independently and then to compare with the set-ups arrived at in that section and to find the reasons for any differences; to plan set-ups for additional problems which suggest themselves to him; and actually to run several such sample problems on the computer long enough to provide a final check of whether the set-ups are adequate.

Additional material on differential analysers may be found in the following references: "The Differential Analyser" by J. Crank, Longmans Green And Company, London, 1947. "A New Type Of Differential Analyser" by V. Bush and S. H. Caldwell, Journal Franklin Institute 240, No. 4, October, 1945. "Applications Of The Differential Analyser" by A. C. Cook and F. I. Maginniss, General Electric Review 52, p. 14, August, 1949. See also "The Differential Analyser, A New Machine For Solving Differential Equations" by V. Bush, Journal Franklin Institute 212, 447, 1931.

II. DESCRIPTION OF COMPUTER

Generally speaking, the computer consists of a collection of independent units mounted together on a rolling table, most of these units being supplied in multiple, for mechanically performing the elementary operations which in combination enable one to solve a system of ordinary differential equations. These units are integrators, multipliers for multiplying any variable by a chosen constant, adders, plotting tables for reading functional relationships graphically out of or into the computer, a motor-driven independent variable unit for turning the independent variable shafts, a hand crank for turning any shaft by hand as in manual curve following; and revolution counters for indicating the numerical value of any chosen variable. While the elementary operations are performed <u>mechanically</u>, it is a general principle of organization of this computer that every input or output variable of every one of the above units is converted from the form of a shaft rotation angle into the form of a three-wire electrical signal and in this latter form is made available at a master plug-board. The conversion of a variable from mechanical to electrical form or vice versa is just the function of the electrical machine known as a synchro[1]

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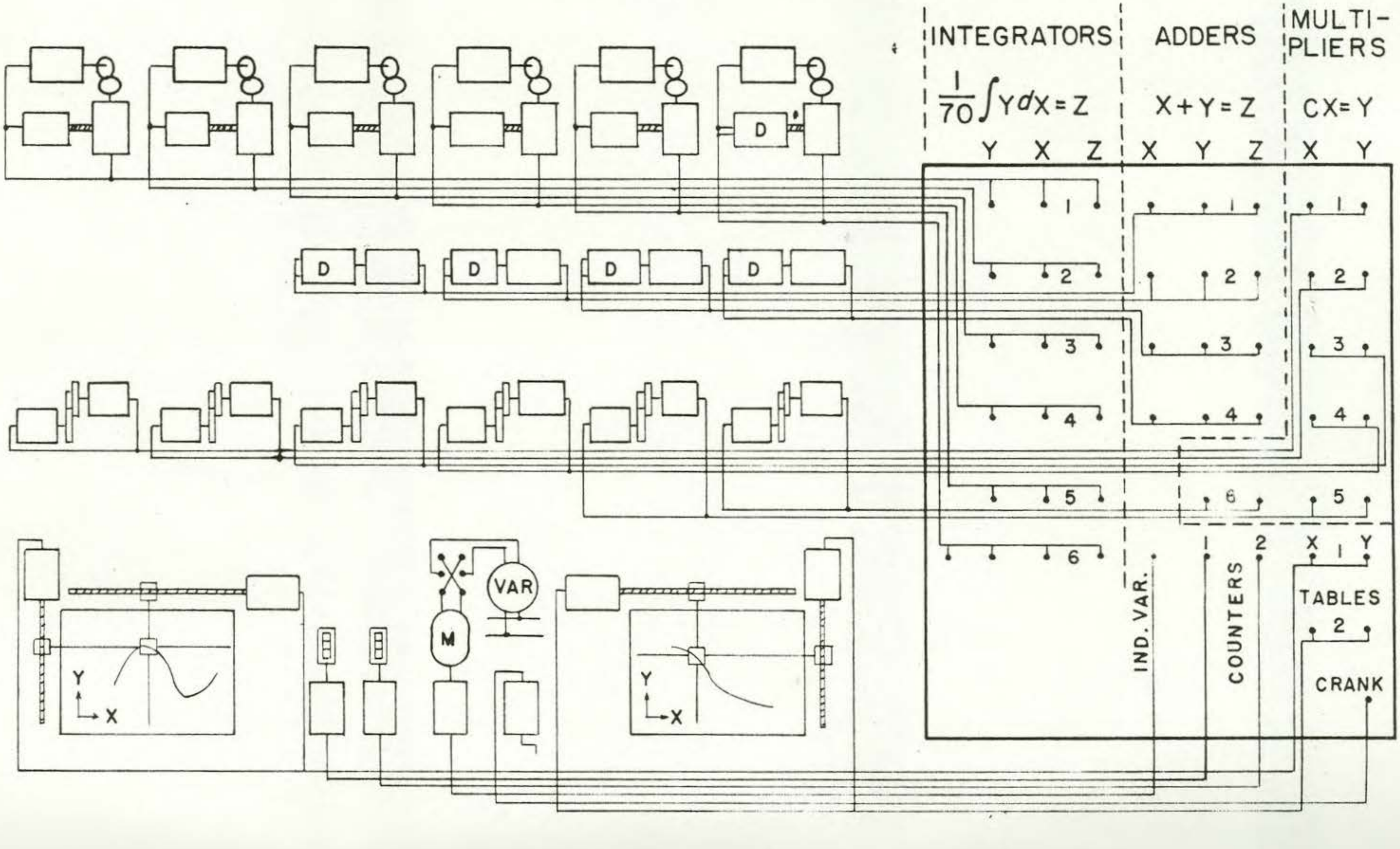
1. See for example "Synchronized Drives Without Mechanical Connection," Electrical Journal, July, 1933; also General Electric Review 33, 500, September, 1930.

or selsyn. No synchro in this computer is permanently connected to any other synchro. Thus, the operation of setting up or clearing out any problem involves only plugging up or clearing a set of connections on the plug board and manually selecting the gear ratios on the multipliers.

Figures 1a, 1b are photographs of the computer and Figure 2 is a simplified diagram showing the logical organization of the instrument. These figures will be referred to and may be studied in conjunction with the following detailed description.

An <u>integrator</u> consists of a flat turntable driven by a synchro, the variable of integration or "X" synchro; and a wheel, the rim of which bears on the turntable and which drives a synchro, the integral or "Z" synchro; and a lead screw driven by a synchro, the integrand or "Y" synchro, which traverses the turntable along a diameter and thus varies the distance of the point of contact between wheel and turntable from the center of the turntable. The integrators in this computer are designed to operate with considerable pressure on the point of contact and with pure rolling motion of the contact. A visible amount of wobbling of the integrating wheels is normal and has no effect on the accuracy of operation. The integrand or "Y" synchro is fitted with a revolution counter to indicate the value of the integrand. Integrator

FIGURE 2
SCHEMATIC OF COMPUTER



Number 1 of the computer shows up most clearly in the left foreground of Figure 1b. The six integrators supplied in this computer appear along the back of the upper level in Figure 1a.

Let the X shaft turn through an angle X, let the Y shaft depart by an angle Y revolutions from its position when the point of contact is exactly centered and let the Z shaft turn through an angle dZ. Then dZ is proportional to Y and to dX; in fact

$$dZ = \frac{P}{R} YdX$$

where P is the pitch of the integrator lead screw and R is the radius of the integrating wheel. In this computer P = .025" and R = 1.750" so that P/R = 1/70, the so-called integrator factor of the computer. Besides the integrator factor, the one other important number pertaining to the integrator is the limit of travel of the lead screw and turntable; this is 50 revolutions on either the positive or the negative side: $-50 \leq Y \leq +50$.

If the X synchro and the Y synchro are driven in a coordinated way by means of connections from other components of the computer, so that Y is a function of X, then we shall have

$$Z = \frac{1}{70} \int_{X_o}^{X} Y(X)dX; \text{ X, Y, Z in revolutions}$$

A multiplier consists of an "X" synchro the shaft of which carries a cone-shaped stack of driving gears of various numbers of teeth and of a "Y" synchro whose shaft carries a driven gear. The driven gear is manually meshed with the chosen driving gear in order to provide the desired multiplying constant C is the relation Y = CX. The values of C available in this computer are 0.1 to 0.9 in steps of 0.1. It is not feasible to operate the multipliers "backwards," i.e. so as to increase the angular speed, because of the accompanying loss of torque, unless C is near unity. The six multipliers appear at the center and left on the lower level in Figure 1a.

An adder consists of a differential synchro, the "X, Y" synchro, which has the property that its shaft assumes an angle equal to the sum (or difference, depending on the polarities of the electrical connections) of the angles represented by the two electrical signals entering it; and of an ordinary synchro, the "Z" or sum synchro. The shafts of the two synchros are direct coupled. The operation of an adder is thus represented by the equation

$$Z = X + Y$$

Four adders are supplied in this computer and are partially visible on the right, on the lower level, in Figure 1a.

A plotting table consists of a plane surface upon which a piece of graph paper or plain paper may be fastened and of a pen or index carriage whose cartesian coordinates

"X" and "Y" are controlled respectively by two synchro driven lead screws. In this computer, these lead screws have 20 threads per inch and the ranges of travel are 140 turns in the ordinate and 200 turns in the abscissa, corresponding to the standard 7" x 10" grid on 8 1/2" x 11" graph paper. The two plotting tables provided in this computer appear on the right and left, upper level, Figure 1a, and one of them appears rather clearly in the right foreground of Figure 1b. A pen is inserted into the index carriage when the computer is to draw a graph on the plotting table in question, while an index is inserted when a previously supplied graph is to be followed manually.

A <u>revolution counter</u> consists of a synchro driving an indicator of its shaft angle. The indicator follows both increases and decreases of shaft angle and is direct reading to 1/10 revolution. The two such units supplied in this computer are located to the right of the row of integrators, Figure 1a.

An <u>independent variable unit</u> consists of an ordinary fractional horsepower reversible motor with manual forward-off-reverse switch and variable speed control, driving a large size synchro. The independent variable synchro is also fitted with a hand crank for manual operation of the independent variable. The one such unit in this computer is located on the right (Figure 1a) half-way between upper and lower level. The switch and speed control for it appear on a vertical panel just under the right hand plotting table.

A hand crank consists of a synchro with a hand crank fitted to its shaft. One such unit is supplied and appears on the vertical panel with the independent variable controls.

The plug-board occupies the right-hand end panel of the computer. On it is an array of identified two-terminal receptacles, one for each synchro (two for each differential synchro) in the whole machine. Two-conductor cords fitted with plugs at both ends are provided for connecting any synchro to any other synchro. The plugs are so constituted that they can be plugged into one another, thus permitting a given driving synchro to control several driven synchros, as is often required.

The several receptacles appearing at the left of the plug board which are of a different nature from the standard synchro receptacles are not involved when this computer is used alone, but only when two such computers are to be used in combination as a double capacity computer. They allow one to tie together several circuits which are common to the whole computer.

The push-button box, visible in the center of the upper level in both Figure 1a and Figure 1b, has the function of allowing the operator to select and individually to drive any synchro shaft which must assume a specified angle, as in adjusting initial conditions. The use of the push button box will be discussed further in section III.

The lamp bank, visible on the left in Figure la, consists of one lamp for each ordinary synchro (none for differential synchros). The lamp belonging to a given synchro normally flashes on as that synchro is plugged to another and then goes out as the pair of synchro shafts settle into a synchronous angular relationship. If the shafts should fail to synchronize properly, the corresponding lamp or lamps remain lighted, thus warning the operator of the faulty condition,and at the same time, reducing the voltage across the primary windings of the affected synchros for overload protection.

Figure 2 is a schematic diagram of the computer. A number of features either common to all the synchros or more or less incidental to the logical functioning of the instrument are omitted from Figure 2 for greater clarity. Among the features not shown are: The supply of 110 volt 60 cycle exciting current to the rotor windings of all ordinary (non-differential) synchros, which supply comes from the A.C. power line through individual series lamps and shunt power factor correcting condensers; a circuit involving a number of limit switches in series, the purpose of which is to warn the operator and stop the independent variable drive when any of the lead screw driven elements nears the mechanical limit of its travel, as in the integrators or plotting tables; the push button circuits referred to above, which are so wired that when a particular button is depressed, it disconnects the independent variable synchro from the plug board and connects it directly and only to the synchro whose shaft position is to be adjusted; a common bus permanently connecting together all No. 3 terminals of the Y-connected information carrying windings of all the synchros (stator windings of ordinary synchros, rotor and stator windings of differential synchros).

The No. 1 and No. 2 terminals of the Y-connected 3-terminal windings of all the synchros are brought to individual two-terminal receptacles on the plug board. These connections are shown in Figure 2, but only schematically in the form of a single line and a single black dot on the plug board, i.e., the No. 1 and No. 2 wires are not shown distinct from one another. Also, the two or three such pairs from an individual unit are shown by a single line in the figure for part of their travel. The reader must, therefore, keep in mind that a line in Figure 2 may represent two, four, six, or eight independent coded conductors. The artifice of connecting all No. 3 synchro terminals together permanently and bringing only Nos. 1 and 2 individually to the plug board allows just the requisite freedom of connection, for it permits any one of three choices: no connection, a connection with a mathematical positive sign by connecting a No. 1 to a No. 1 terminal and No. 2 to No. 2, or a connection with a mathematical negative sign by connecting No. 1 to No. 2 and No. 2 to No. 1. The cords and receptacles in this computer are so wired that if the cord leaves both plugs downward (or upward) a mathematical positive connection exists, while if the cord leaves one plug downward and the other upward, a negative connection exists.

With the explanations of the two preceding paragraphs (which explain mostly what does not appear in Figure 2!) and the preceding material, Figure 2 should be clear to the reader. Receptacles appearing on the actual plug board but not in Figure 2 are not in use as of this writing. Integrator Number 6 in the actual computer has a differential synchro for its integrand or Y synchro so that it can add two variables and integrate the sum with respect to any chosen variable. It may also be used normally to integrate one variable by supplying that variable to one of its Y input receptacles and locking the other. This special feature of integrator No. 6 is shown in Figure 2.

We complete this section with several items of miscellaneous information. All the revolution counters read directly in tenths of a shaft revolution. The precision of the computations occasionally warrants estimating the nearest twentieth of a revolution, but hardly ever anything closer than this. The counters on the integrand synchros read zero for zero integrand, read the actual shaft angle in tenth revolutions when the integrand is positive, and read the shaft angle plus 100.0 revolutions when the integrand is negative. E.g. - 37.3 revolutions is indicated by a counter reading of 62.7.

The lead screws for the plotting tables are coupled to the driving synchros through friction clutches which do not slip in normal operation but which may be manually slipped in order to secure a fine adjustment of the position of the pen or index on the paper. To make such a fine adjustment, one holds the knurled aluminum wheel adjacent to the synchro and forcibly turns the knurled aluminum wheel adjacent to the lead screw. A coarse adjustment of the position of the pen or index can be made most rapidly by disengaging the half-nut in the carriage block driven by the lead screw and moving the carriage block manually to the desired location. To disengage the half-nut, one grasps the knurled brass button on top of the carriage block and pulls it up about 1/8 inch relative to the block. The block itself should not be lifted off its track in this manipulation. After shifting the carriage block to the desired location and releasing the knurled brass button, the operator must see that the half-nut is again properly engaged with the lead screw.

III. ROUTINE FOR SETTING UP AND OPERATING COMPUTER

In this section, we describe a detailed step by step procedure for setting up and running the machine. This orderly routine should be followed in detail for greatest efficiency of use of the computer. It is assumed that a plug-board connection chart of the type shown on page 36 for the problem to be solved, has been worked out. The working out of this chart will be discussed in the next following section.

Step 1. Remove all cords from plug board. The last previous user will often have left his setup on the plug board, and this is regarded as legitimate practice since the previous user may be the next user, in which case he saves the labor of replugging his setup, and since removing the cords from the plug board takes much less time than installing them correctly.

Step 2. Plug the power cord of the computer into a 115 volt 60 cycle wall receptacle and turn on the power switch. This energizes all the ordinary (non-differential) synchros in the computer and must be done before plugging at the plug board so that the synchro linkages will synchronize one by one as they are connected. See that the independent variable switch is in the off position. This insures that the integrating surfaces are disengaged and will not be damaged by a forcible rotation which occurs during synchronization of a pair of synchros or by sliding action while a turntable is being traversed for purposes of adjusting the initial value of the integrand as in Step 5 below.

Step 3. Clip the plug board connection chart for the problem to be solved above the plug board and plug up the connections according to it. This operation must be performed systematically because the tangled mass of wires which will exist on the plug board is difficult to trace. A good system is to start with the lowermost horizontal line on the chart and with the (single) source of torque (driving element) for the variable represented by that line and to plug from the driving element to any driven element indicated as connected to that line on the chart; then to any other driven element, etc., until all the connections called for on the lowest line of the chart are made. Then, proceed to the line next to the lowest and make the connections indicated by it in the same order, starting with the source of torque for the corresponding variable, and so forth until all connections indicated on the chart are made. The connections for each line should be checked, including signs, before proceeding since they would be difficult to check later.

As each synchro is connected to another in the above process, the lamps in the lamp bank corresponding to these synchros will usually flash on, indicating a temporary condition of nonsynchronism between the pair of shafts involved, and will then go out, indicating that the shafts have settled into synchronism. If the lamps remain lighted, this is usually a symptom of a "dead center" condition and this condition can usually be remedied by temporarily inverting one plug so as to kick one of the shafts to a different angle and then reinserting the plug in the originally required orientation. If this artifice does not work, one may gain access to either of the shafts for purposes of shifting its position by plugging the hand crank to it. If the lamps still refuse to go out, either a conflicting set of connections or a fault in the wiring of the computer is indicated. The lamps may not flash at all when a given connection is made because the pair of shafts involved may happen to be near synchronism initially. In this case, temporarily inverting a plug will usually reassure the operator that he is not dealing with an open circuit condition. In any case, the operator should be sure that synchronism is established in any given connection before proceeding to make further connections.

The plugging operation is unfortunately accompanied by an electric shock hazard (though hardly a dangerous one), since once one end of a cord is plugged in, the prongs of the free plug may have up to 105 volts of potential difference between them. Hence, the operator should hold the live plug in such a way that the prongs do not touch him or anyone else or any metal.

Step 4. Set the correct gear ratio on each multiplier being used in the problem, as indicated on the connection chart. Be sure that the gears are properly meshed and free to turn. This operation is also accompanied by a shock hazard in the present computer because of exposed electrical connections on the synchro which must be moved manually. Therefore, the synchro should be grasped by the cylindrical body portion without touching the live end portion.

Step 5. Set the initial conditions on the integrands, on the plotting tables and on the free revolution counters if they are in use. This is done as follows: Turn the independent variable speed control knob up to some intermediate value, roughly 90 volts on the dial or whatever setting gives the desired speed in the operator's experience; depress the push button corresponding to the variable to be adjusted; operate the forward-off-reverse lever on the push button box until the revolution counter in question shows the desired reading. If in this process the operator experiences difficulty with overshooting the desired setting under electric drive, he may always resort to manual adjustment of the shaft angle by operating the hand crank on the independent variable synchro. Then proceed to the next variable to be adjusted, etc.

Several complications may occur in this process of adjusting initial conditions. The independent variable motor may, for example, refuse to operate because one or more limit switches are open. In this case, the offending variable or variables must be brought within allowable limits manually. If the offending variable is an integrand, depress the corresponding push button and turn the independent variable synchro manually by means of the crank fitted to it. Be certain to turn the crank in the proper sense to bring the variable within limits rather than to drive it farther out of limits; otherwise, mechanical damage to the integrator may result or the lead screw may disengage entirely.

Another complication which may occur is that the variable to be adjusted may refuse to move because it is connected directly or indirectly to the output of a multiplier set to a low gear ratio. In this case, the attempt to adjust the variable involves attempting to drive a low gear ratio multiplier backwards, and the synchro linkages cannot supply enough torque for this purpose. The remedy is to identify which multiplier is causing the difficulty, disengage the gears of that multiplier entirely, adjust the variable in question, and then reengage the multiplier gears.

Still another complication occurs when two variables require to be independently adjusted to prescribed values but are connected together at the plug board. The remedy in this case is temporarily to disconnect them at the plug board, adjust their values and reconnect. In reconnecting, one will usually displace one or both slightly from the prescribed initial values and a second adjustment operation will be required.

The integrand synchros are so adjusted that when any pair of them is synchronized the counter readings will differ by a whole number of revolutions. This means that if two integrands are required by the problem to be the same variable except for a constant difference, the constant difference cannot be arbitrarily chosen in this computer but must be a whole number of turns. However, since one turn usually represents quite a small increment of the mathematical variable, this circumstance has not been found to be inconvenient. This restriction does not apply to integrand synchros which are differential synchros as in integrator No. 6 in the present computer.

After each initial value has been individually attended to, it is important to check all of them again since the process of setting the initial value of one variable may displace a previously set variable.

<u>Step 6</u>. Make a fine adjustment of the position of the carriage on each plotting table using the index in the

carriage and manipulating the friction clutches described at the end of Section II. Then replace the index with a pen (check whether the pen will write) if the machine is to draw a curve or leave the index if a curve is to be followed.

Step 7. Turn the variac speed control back to zero, then turn the independent variable forward-off-reverse switch to the forward position, then turn up the speed control to give the desired speed of operation. These operations must be performed in this order because the integrating wheels are engaged upon turning the switch to forward (or reverse) and the wheels should be engaged before the shafts start turning. If the shafts were to start turning before the wheels were fully engaged, an uncontrollable error would be introduced in the initial conditions. Do not operate the forward-off-reverse switch unless both the speed control knob is at zero and all shafts are at rest.

High speed and high acceleration are detrimental to accuracy. In general, the synchro shafts should turn no faster than about 300 r.p.m., and the machine should be accelerated and decelerated gently, because otherwise the considerable inertia of the synchro rotors increases the follow-up errors in the synchro linkages.

An operator who is following a curve by means of the hand crank should have the speed control available to him so that he can slow down the operation for accurate following of portions of the curve with large slope or large curvature.

Step 8. Stop the machine by operating the speed control (not the switch) when the desired amount of solution of the problem has been generated. If the criterion for "end of run" is that a certain variable take on a prescribed value, the operation must be slowed down before that value is attained; otherwise, the rotor inertia will cause the machine to over-ride the end point. If this happens, the solution may be retraced (with some slight loss in accuracy) by first turning the speed control to zero; then after the shafts have come to rest, throwing the switch to the reverse position, then manipulating the speed control.

In fact, a solution curve may be retraced all the way in order to provide reassurance that no slippage or other malfunction was present.

This completes the step-by-step description of how to operate the computer. Several comments are in order here concerning the actual running. The machine may be left running unattended, provided the operator is sure from previous experience or from the nature of the problem, that it will not run off limits at high speed; if it runs off limits at moderate speed, it will stop itself safely. The limit switches on the integrators are adjusted so that they cut off the power to the independent variable drive approximately four lead screw revolutions before the mechanical limits, and unless the integrand synchro is turning quite fast, it will not be carried so far by inertia.

If an integrand variable runs off limits and its rate of change is large, the problem setup must be revised with more appropriate scale factors and run over again. If an integrand variable runs off limits but its rate of change is sufficiently small and decreasing, it is often possible to save the solution by cranking the independent variable by hand until the variable comes back within limits; however, no integrand should ever be taken manually beyond the extreme limits of $\pm$ 51.0 revolutions.

If too large a scale has been chosen for displaying a solution graphically on a plotting table, the solution need not be given up since one may readily "fold the scale" by stopping the machine, removing the pen, shifting the pen carriage some suitable distance and reengaging, reinserting the pen and continuing.

IV. PLANNING THE SETUP

The problem to be solved will in general involve an ordinary differential equation or a system of simultaneous ordinary differential equations together with sufficient initial conditions to determine a solution. The following explanation of procedure is meant to be quite general but will, nevertheless, be illustrated at every step for the sake of concreteness by means of a particular sample problem.

The sample problem which we shall treat is:

$$\frac{d^2x}{dt^2} = f(x) \sin t$$

where $f(x)$ is given in the form of a graph or of a numerical table from which a graph may be constructed. $f(x)$ varies between the limits ± 1. We are given $x = 0$ and values of $\frac{dx}{dt}$ in the range 0 to 0.3 at $t = 0$ and are required to find the values of x and of $\frac{dx}{dt}$ after four cycles of the sine function, i.e., at $t = 8\pi$. If in any solution x returns to 0, the solution is to be stopped at that point. Each pair of initial values x_o and $\left(\frac{dx}{dt}\right)_o$ gives a distinct solution, of course; but we want, if possible, a single setup which will handle all solution curves corresponding to all initial conditions in the range specified.

Incidentally, the sample problem we have chosen is non-linear and is not analytically soluble except in the trivial case $f(x)$ = constant.

<u>Step 1</u>. Break the equation or equations down into a system of simultaneous first order equations by introducing auxiliary dependent variables, and solve for the (first) derivatives. In our example, this comes to introducing $v = dx/dt$ so that we have:

$$\frac{dv}{dt} = f(x) \sin t \qquad (1)$$

$$\frac{dx}{dt} = v \qquad (2)$$

Step 2. Integrate each of these equations formally with respect to the independent variable:

$$v = \int f(x) \sin t \, dt \qquad (1')$$

$$x = \int v \, dt \qquad (2')$$

The integrals indicated are definite integrals with fixed lower limits and variable upper limits. However, we need not concern ourselves with the values of these limits of integration at this point in the procedure since they will automatically be put into the computer correctly in Steps 5 and 6 of Section III.

Step 3. The right hand members of the resulting equations will in general involve sums of variables, products of variables by constants, products of pairs of variables, and ratios of variables. The sums of variables and products of variables by constants may be passed over since they can be handled by the adders, respectively multipliers, in the computer. Products of two variables may be handled in one of two ways: Suppose the product in question to be r s; then we introduce a further auxiliary dependent variable q = r s and calculate q from the equation

$$q = \int r \, ds + \int s \, dr$$

which will use up two integrators and an adder and will provide r s given r and s. The second way is applicable if the product r s itself is not required but only the integral of the product with respect to any further variable, say t: Introduce an auxiliary variable p as:

$$p = \int s \, dt$$

$$\int r \, s \, dt = \int r \, dp$$

which provides $\int r \, s \, dt$ at the cost of two integrators.

With regard to ratios of variables, two methods are available on the present computer, neither method very satisfactory. Suppose that the ratio to be formed is r/s; we may form the reciprocal of s by introducing auxiliary variables $w = \frac{1}{s}$ and $u = w^2$ and using the equations

$$u = 1/2 \int w \, d \, w$$

$$w = - \int u \, d \, s$$

which use up two integrators and provide $\frac{1}{s}$, given s. The formation of a ratio of variables is thus reduced to the formation of a product, which was treated above. A disadvantage of this method is that $u = \frac{1}{s^2}$ occurs as an integrand and gets very small for large s, with adverse effects on the accuracy. Another disadvantage of the method is that it requires a total of four integrators.

The second available method is applicable only if the ratio r/s itself is not required but only the integral of the ratio with respect to some further variable t and if further s varies over only a small range percentage-wise. We form

$$p = \int r \, dt$$

by means of an integrator and place s as integrand on a second integrator and drive the integral synchro of the second integrator with p and take the result off the variable of integration synchro:

$$\int \frac{r}{s}\, dt = \int \frac{dp}{s}$$

The difficulties and restrictions mentioned in the last two paragraphs are all related to the appearance of a variable in a denominator, i.e. to division by a variable. In general, the computer as presently constituted can perform certain mathematical operations readily while the inverse operations are difficult or impossible. Multiplication, respectively division, by a variable is an example; other examples are multiplication by a constant < 1, respectively multiplication by a constant > 1; and integration, respectively differentiation. We plan eventually to supply an additional type of unit in the computer, known as a servo unit, which will have the general capability of making available the inverse of any mathematical operation which is itself available in the machine.

Returning to our sample problem after this digression, we see that the only complication of the type described in Step 3 is the product of f(x) by sin t, this product to be integrated with respect to t. According to the recipe given above, we may form either the auxiliary variable $p = \int f(x)\, dt$ or the auxiliary variable $q = \int \sin t\, dt = -\cos t$. In the next step, we shall see that cos t has to be computed in any case for other reasons; hence, for economy of integrators, we choose the second alternative and write (1') in the form:

$$v = -\int f(x)\, d \cos t \qquad (1'')$$

Step 4. The right hand members of the equations may involve functions of variables and any such functions must be computed or otherwise provided. This may be accomplished by placing a graph of the functional relationship on a plotting table and arranging to follow the curve manually (this is how $f(x)$ is to be treated); or by performing algebraic operations as described in Step 3 if we are dealing with a rational function; or by solving an auxiliary system of differential equations obeyed by the function to be supplied.

$f(x)$ is to be supplied from a previously prepared graph, followed manually; hence, no additions or modifications of the working equations are required for $f(x)$.

cos t will be supplied by solving the two additional working equations:

$$\cos t = -\int \sin t \, dt \qquad (3)$$
$$\sin t = \int \cos t \, dt \qquad (4)$$

This requires two additional integrators. We now recognize that the sample problem requires a total of four integrators for its solution.

Step 5. Choice Of Scale Factors

All of the variables occurring in the four working equations must now be put into correspondence with variables occurring in the machine, the latter variables being naturally measured in shaft revolutions. We have been using the convention of designating natural machine variables measured in shaft revolutions by capital letters and mathematical variables by lower case letters. Each mathematical variable is proportional to one or more machine variables, and the constants of proportionality are called scale factors. For example, we shall find that in our sample problem $70\ t = T$, where t is the independent variable of the mathematical problem, T is the shaft angle of the independent variable synchro, in revolutions, and 70 is the scale factor. Sometimes more than one machine variable corresponds to a single mathematical variable because more than one scale factor is required for that variable in setting up the problem. For example, the increment dt occurring in (2') and the increment dt occurring in (3) might have to be represented at different scale factors.

The choice of scale factors is in general governed by the following rules:

a) The scale factors of all variables occuring in any one working equation must balance in the sense that the machine equation is equivalent to the working equation. E.g. suppose that integrator No. 1 is assigned to working equation (1"). The two equations which must be equivalent are:

working equation: $v = -\int f(x)\, d \cos t$

machine equation: $Z_1 = \frac{1}{70} \quad Y_1\, d\, X_1$

A set of scale factors satisfying the requirement would then be given by:

$$X_1 = -\ 35 \cos t$$

$$Y_1 = 60\ f(x)$$

$$Z_1 = 30\ v$$

The scale factors of variables appearing in adder or multiplier equations must of course also balance in the same sense.

b) Any variable appearing as an integrand must appear at such a scale factor that the machine variable remains within the limits $\pm$ 50 revolutions and should for maximum accuracy appear at a scale factor which uses up a considerable portion of this range. (The same mathematical variable may of course appear at other scale factors elsewhere in the machine). The example given under a) violates this rule since $-1 \leq f(x) \leq 1$ and the integrand Y_1 has been put equal to 60 f(x). A revised choice might be:

$$X_1 = -\ 35 \cos t$$

$$Y_1 = 50\ f(x)$$

$$Z_1 = 25\ v$$

c) Any variable appearing as an ordinate or abscissa on a plotting table must appear at so small a scale factor as to remain within the mechanical limits of travel of the carriage (unless it is planned to fold the scale on the graph paper) and on the other hand, at so large a scale factor as to make reasonably complete use of the graph paper surface. For example, suppose that plotting Table 2 (which is near the variable speed control) is assigned to the graph of $f(x)$ versus x. Then we might put

$$Y_2 = 50\ f(x)$$

$$X_2 = 2\ x$$

The first of these scale factors makes the graph of $f(x)$ cover 100 revolutions total vertically, or 5 inches, which is reasonable. The second scale factor allows for an anticipated maximum value of x equal to 100. This latter number is arrived at as follows: The largest possible value of d^2x/dt^2 is $|\sin t|$, the average of which is $2/\pi$. Thus, if $f(x)$ is such that for some initial conditions this largest possible value is nearly attained for all t, we shall have

$$x_{max} \cong \frac{1}{2}\ \frac{2}{\pi}\,(t_{max})^2 = 64\pi$$

The choice of scale factor here then depends on the form of the function f and our choice is based on the largest actually occurring x being about half the maximum possible x.

These rules a), b), and c) do not determine the scale factors for a given problem uniquely but merely require them to satisfy a complicated set of equalities and inequalities. Rather than to write down all such equalities and inequalities and to attempt to solve them analytically, it has been found much better to proceed by a kind of trial and error method in order to find a workable set of scale factors. We shall try to illustrate this trial and error method on the sample problem. Note that the integrator factor (70 in our machine) and the limits of travel are important in determining the scale factors. In building this computer, 70 was chosen as integrator factor for mechanical reasons; this was not a fortunate choice since a convenient integrator factor is one having many different small prime factors, such as 60.

We have already made some tentative assignments of scale factors in the sample problem, namely:

$$25\ v = -\frac{1}{70}\int (50\ f(x))\ d(35 \cos t) \qquad (1'')$$

Proceeding further, we may tentatively assign scale factors for the variables in equation (2'). Here we find v as integrand, so that the scale factor for it depends on its largest anticipated value. By the same argument as used above for x_{max}, we estimate

$$v_{max} \cong \frac{1}{2} \cdot \frac{2}{\pi} \cdot 8\pi = 8$$

Therefore, we assign the following scale factors for equation (2'):

$$5\ x = \frac{1}{70}\int (5v)\ d(70\ t) \qquad (2')$$

The scale factor 5 for v provides reasonable assurance the integrand will not go off scale and also bears a simple relation to the other scale factor for v, namely 25, so that a multiplier set at 0.2 sufficies. 5x can likewise be converted into 2x as required on the plotting table, by a multiplier set at 0.4. 70t is also the machine variable convenient for generating cos t, as we shall see immediately.

Equations 3 and 4 may then have tentative scale factors introduced for them as follows:

$$35 \cos t = -\frac{1}{70}\int (35 \sin t)\, d(70\, t) \qquad (3)$$

$$35 \sin t = \frac{1}{70}\int (35 \cos t)\, d(70\, t) \qquad (4)$$

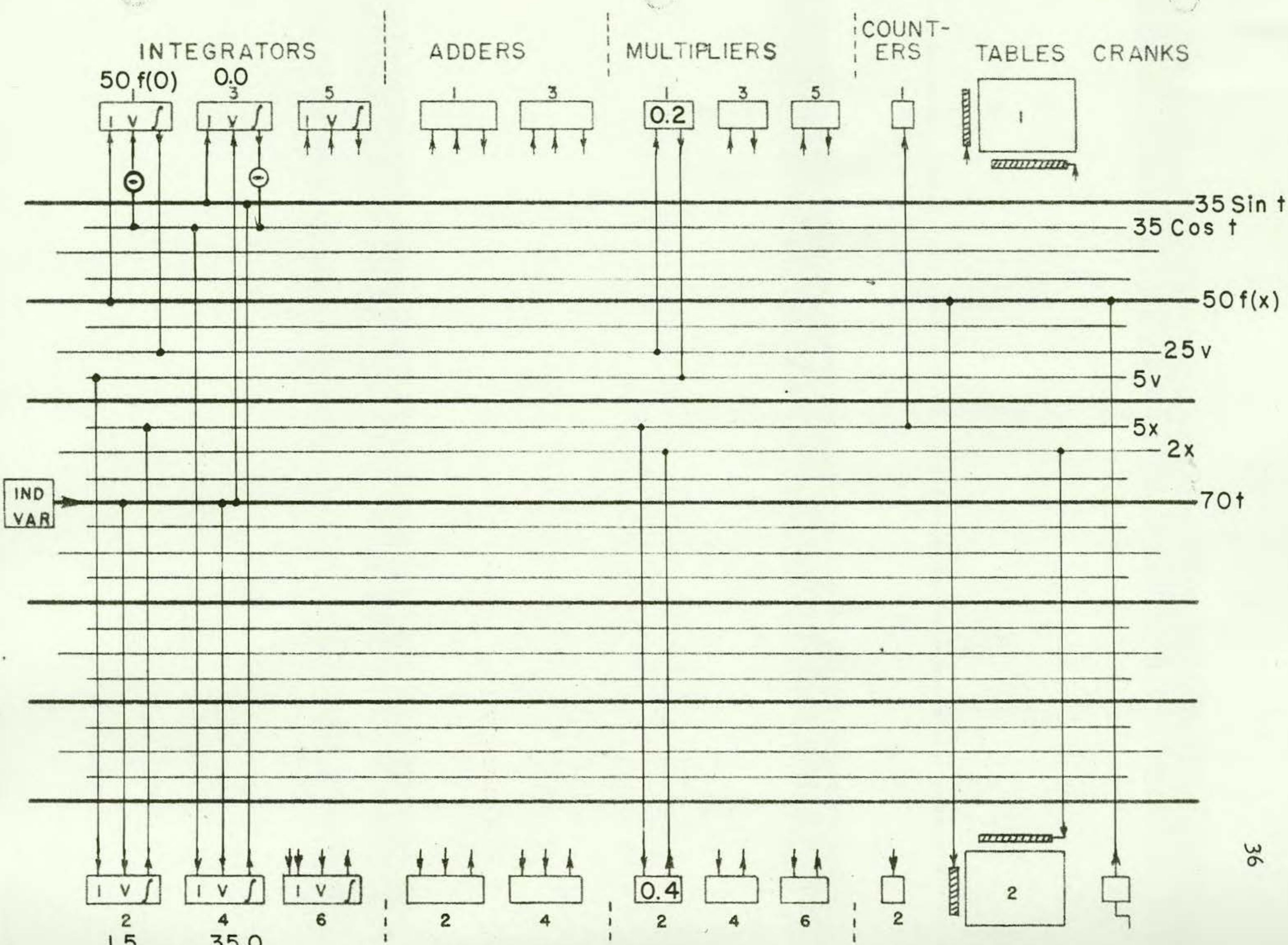
The factor 35 is chosen for cos t since 35 cos t is what is required for equation (1"). All the remaining scale factors in equations (3) and (4) follow quite naturally.

The ultimate results required in the sample problem were the values of x and of $dx/dt = v$ at $t = 8\pi$. It is possible to read 5v directly off the integrand revolution counter of the integrator assigned to equation (2'). x may be read from the position of the index on the graph of f(x), or if the value read off the graph is not precise enough, 5x may be connected to one of the free revolution counters and read off it. In either case, all the scale factor requirements have been satisfied tentatively. The next step in solving the sample problem would be to set it up on the computer with these scale factors and to observe the actual range of variation of the integrand 5v and the actual range of the abscissa 2x. If either of these ranges turned out to be impossibly large or inconveniently small, a revision of the scale factor assignments would be required.

As a general rule in scale factor assignment, it is important to have all the scale factors for any one mathematical variable bear simple ratios to one another so that the multipliers may be used readily; and to have the least possible multiplicity of scale factors because of the limited supply of multipliers. Occasionally, an integrator which is not otherwise assigned may be used as a multiplier. When used in this manner, its integrand Y is set as such a value that Y/70 is the desired multiplying factor and is left unconnected at the plug board.

Step 6. Making Plug-Board Connection Chart

This chart is merely an orderly record of the machine problem including scale factors, definite assignments of elements in the computer to specific tasks, and initial values of all the variables, in a form convenient for the plugging-up operation. Such a chart is necessary in all except the very simplest problems in order to avoid confusion. Blanks for constructing such charts are available and a small supply of them is included in this manual. The following page is the connection chart for the sample problem.



DIFFERENTIAL ANALYSER CONNECTION CHART

The blank has symbols for each element available in the machine and a set of horizontal lines, every fourth line accentuated and lengthened in order to aid the eye in following the connections. The lines provide a means of representing the connections in an orderly readable fashion. Each line which is used in making connections must have one and only one source of torque entering it and may have an arbitrary number of torque loads connected to it. Arrows in the chart indicate the sense of flow of torque. Any horizontal line in use is labeled at either end with the value of the machine variable which it carries, in shaft revolutions. Negative connections are symbolized by a negative sign in a circle, inserted into the appropriate line; see the connection to the variable of integration of integrator 1, page 36. Initial values of integrands may be written adjacent to the appropriate integrator, in units of shaft revolutions since the revolution counters read thus. The proper settings of multipliers are written into the center of the multiplier symbol.

V. SETUPS OF ADDITIONAL SAMPLE PROBLEMS

1. $\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + x = 0 \quad \alpha = -0.2, +0.1, +0.5, +1.0;$

$x = 0$ and $\frac{dx}{dt} = 1$ at $t = 0$. Plot both x vs. t and v vs. x.

Working equations with scale factors:

$$35x = \frac{1}{70}\int (35v)\, d(70t) \qquad (1)$$

$$35v = -\ \alpha(35x) - \frac{1}{70}\int (35x)\, d(70t) \qquad (2)$$

Connection chart see page 46

2. $\frac{d}{dt}\left[f\left(\frac{dx}{dt}\right)\ \frac{dx}{dt}\right] + x = 0$ (L-C circuit with non-linear inductance)

$f(v)$ is an even function, $f(o) = 1$ and $f(v) \rightarrow 0.1$ as $v \rightarrow \infty$.

We are to plot x vs. t. $x = 0$ and $\frac{dx}{dt} = 0.2,\ 0.5,\ 1.0,\ 1.5$ at $t = 0$.

Let $\varphi = f(v)v$ and prepare a graph of φ vs. v.

The working equations with scale factors may be taken to be:

$$40\,\varphi = -\frac{1}{70}\int (40x)\ d\ (70t) \qquad (1)$$

$$40\,x = \frac{1}{70}\int (28v)\ d\ (100t) \qquad (2)$$

Connection chart see page 47

3. To draw a Cornu Spiral.

The computer is to plot y vs. x, where the relation between y and x is given parametrically by

$$y = \int_o^t \sin t^2\ dt \qquad x = \int_o^t \cos t^2\ dt$$

Assume that ten complete turns of the spiral are to be drawn. Then the range of t^2 is $0 < t^2 < 20\pi$ and the range of t is approximately $0 < t < 8$. The working equations with scale factors may be:

$$70t^2 = \frac{1}{70} \int (4.9t)\, d\,(2000\ t) \qquad (1)$$

$$42 \cos t^2 = -\frac{1}{70} \int (42 \sin t^2)\, d\,(70\ t^2) \qquad (2)$$

$$42 \sin t^2 = \frac{1}{70} \int (42 \cos t^2)\, d\,(70\ t^2) \qquad (3)$$

$$60\ y = \frac{1}{70} \int (42 \sin t^2)\, d\,(100\ t) \qquad (4)$$

$$60\ x = \frac{1}{70} \int (42 \cos t^2)\, d\,(100\ t) \qquad (5)$$

For the connection chart, see page 48. Note that rather lavish use of multipliers is made here. The reader may attempt to revise the plan so as to use less multipliers.

4. The driven non-linear oscillator of van der Pol:

$$\frac{d^2x}{dt^2} + \epsilon(x^2 - 1)\frac{dx}{dt} + x = a \cos \omega\, t$$

We are to treat the case $\epsilon = 1$, $a = 1$, $\omega = 0.8$ and are to find the waveform of $x(t)$ and of $v(t)$ after the solution has become steady and periodic.

This is a definitely more complicated problem than any of the previous ones and the determination of a good set of scale factors is more difficult; therefore, we shall discuss the process in more detail.

The working equations before introduction of scale factors may be:

$$v = -\int (x^2 - 1)\, dx - \int x\, dt + \frac{1}{0.8} \sin 0.8t \qquad (1)$$

$$x = \int v\, dt \qquad (2)$$

$$x^2 = 2 \int x\, dx \qquad (3)$$

$$\cos 0.8\,t = -\int \sin 0.8t\, d(0.8t) \qquad (4)$$

$$\sin 0.8\,t = \int \cos 0.8t\, d(0.8t) \qquad (5)$$

Now v and x both occur as integrands so that an estimate of their extreme values must be made. If x $= \pm 2$, the damping coefficient $\epsilon(x^2 - 1)$ is large and positive; therefore, we may reasonably expect that x will not get very far outside the range ± 2. The wave form of x may be expected to be appreciably different from sinusoidal; hence, the extreme values of v may be greater than those of x; assume that v will vary between approximate limits ± 3. Tentative values for the corresponding integrands in revolutions might then be 12v and 20x or 14v and 21x. Note that these two scale factors probably should both or neither contain a factor 7 because $\int x\, dt$ is part of the expression for v and $\int v\, dt$ is x.

$(1 - x^2)$ is another integrand and its limiting value is about $- 3$. A scale factor of 10.5 will probably do for this variable. From equation (3), it is clear that if the scale factor for x^2 is to have a factor 7, those for x and dx should contain 7. Therefore, we shall try the second alternative suggested in the previous paragraph.

The working equations with scale factors will now read:

$$14v = -\frac{1}{70}\int 10.5(x^2 - 1)\, d\left(\frac{280}{3}x\right) - \frac{1}{70}\int (21x)\, d\left(\frac{140}{3}t\right)$$

$$+ \frac{14}{0.8}\sin 0.8t \qquad (1)$$

$$\frac{280}{3}\, x = \frac{1}{70}\int (14v)\, d\left(\frac{1400}{3}\, t\right) \qquad (2)$$

$$10.5x^2 = \frac{1}{70}\int (21x)\, d(70x) \qquad (3)$$

$$\frac{28}{0.8} \cos 0.8t = - \frac{1}{70} \int \frac{28}{0.8} \sin 0.8t \; d(56t) \qquad (4)$$

$$\frac{28}{0.8} \sin 0.8t = \frac{1}{70} \int \frac{28}{0.8} \cos 0.8t \; d(56t) \qquad (5)$$

Note that all the scale factors are balanced in each equation and that we have arranged to generate every variable at a scale factor equal to or greater than any required scale factor for that variable. In equations (4) and (5) we have arranged to generate sin 0.8t at twice the scale factor at which it is required in order to improve the accuracy by using more of the mechanical range of the integrators involved. This costs one multiplier; if no multiplier is available for the task, we have to revise equations (4) and (5) accordingly. A multiplier does not reduce the accuracy at all since the lost motion in a pair of meshed gears is negligible and is not cumulative.

The next consideration is whether the changes of scale factor required according to the above tentative setup are possible with the available multipliers. The independent variable synchro will generate $1400t/3$; t is required at the additional scale factors $140t/3$ and $56t$. These operations require 3 multipliers since:

$$\frac{1400}{3} \times 0.2 = \frac{280}{3}; \; \frac{280}{3} \times 0.5 = \frac{140}{3}; \; \frac{280}{3} \times 0.6 = 56$$

Next x is generated at the scale factor $\frac{280}{3}$ and must also be supplied at the scale factors 70 and 21. These operations require 3 multipliers, one of them set at 0.8 and operated backwards, which is permissible although not the best practice:

$$\frac{280}{3} \times 0.6 = 56; \quad 56 \times \frac{1}{0.8} = 70; \quad 70 \times 0.3 = 21$$

This uses up all the six multipliers available on one table. If no additional multipliers are available, the scale factors 28/0.8 in equations (4) and (5) must be reduced to 14/0.8.

The scheme we have now arrived at is workable but can be improved. The main difficulty with these scale factors is the occurrence of so many 3's in the denominators, and this can be traced back to the ratio 2/3 between the scale factor for v and that for x. Let us try equal scale factors 14 for both these variables; this uses about 60 per cent of the range of the integrators with x as integrand but frees several multipliers and enables one to use the range of the sine and cosine integrators more effectively. Equation (1) will now read:

$$14v = -\frac{1}{70}\int 14(x^2 - 1)\, d(70x) - \frac{1}{70}\int (14x)\, d(70t) + \frac{14}{0.8}\sin 0.8t \tag{1}$$

or better, in order to insure that the first integrand stays within bounds:

$$14v = -\frac{1}{70}\int 14(x^2 - 2)\, d(70x) - 14X - \frac{1}{70}\int (14X) d(70t) + \frac{14}{0.8}\sin 0.8t \tag{1}$$

The rest of the equations will now read:

$$140x = \frac{1}{70}\int (14v)\, d(700t) \tag{2}$$

$$14x^2 = \frac{1}{70}\int (14x)\ d(140x) \qquad (3)$$

$$\frac{14}{0.32}\cos 0.8t = -\frac{1}{70}\int \frac{14}{0.32}\sin (0.8t)\ d(56t) \qquad (4)$$

$$\frac{14}{0.32}\sin 0.8t = \frac{1}{70}\int \frac{14}{0.32}\cos (0.8t)\ d(56t) \qquad (5)$$

The various scale factors for t require 2 multipliers, those for x require 2 and another may be used to reduce (14/0.32) sin 0.8t to (14/0.8) sin 0.8t.

This last set of scale factors may be regarded as satisfactory for this problem.

In order to determine when the wave form becomes periodic, it is best to plot v vs. x on one plotting table, and starting with any reasonable initial values of v and x, operate until the v vs. x curve becomes a closed loop. When this condition has been reached, x may be plotted against t on one plotting table and v vs. t on the other. If the phase relations are important, the forcing term may afterwards be plotted on the same paper in the correct relative phase.

For the connection chart see, page 49.

5. $\frac{d^2}{dt^2}x(t) + r\frac{d}{dt}x(t) + x(t-1) = 0.$

We want to know how large r must be in order to make the solution of the equation stable. Values of r equal to 1.0, 1.2, 1.4 are to be investigated initially. The straight-forward way to do this problem would be to let the machine plot x(t) vs. t and simultaneously to follow the curve manually at the abscissa corresponding to (t - 1) in order to feed the last term into the equation. We cannot do this with the present machine because we are not equipped to draw a curve and follow a curve on the same plotting table at the same time. Therefore, we shall arrange to use two plotting tables and alternate them between the plotting and following tasks.

The working equations with scale factors are relatively simple. Since the equation is linear and homogeneous in x, we can use scale factors 1 for x and for v:

$$v = - r\,x - \frac{1}{70}\int x(t - 1)\; d(70t) \qquad (1)$$

$$2x = \frac{1}{70}\int v\; d(140t) \qquad (2)$$

Multipliers required are: One for changing from 140 to 70t, a second for changing from 2x to x and a third for changing from 2x to rx (except in the case r = 1).

The connection chart is given on page 50. We operate for one unit of t, as indicated on rev. counter 1, with the connections represented by solid lines, then stop and shift to the dotted connections, operate for one more unit of t, stop and shift back to the solid connections, etc. Some curve of x vs. t for the interval $-1 < t < 0$ must obviously be supplied in order to get the process started.

6. Fourier Analysis

We are given a function f(t) in graphical form in the range $0 \leq t \leq 2\pi$ and are required to find its fourier coefficients

$$a_n = \int_0^{2\pi} f(t) \cos n\, t\, dt$$

$$b_n = \int_0^{2\pi} f(t) \sin n\, t\, dt$$

With the equipment available on one 6-integrator unit, it is almost but not quite possible to calculate four coefficients at once. The additional equipment needed is one more hand crank and two more free revolution counters. Assume that this rather minor additional equipment is available. Then the working equations for calculating the first four coefficients, with scale factors (supposing $-1 < f(t) < 1$), may be:

$$70 \cos t = -\frac{1}{70} \int 35 \sin t \; d(140t) \qquad (1)$$

$$70 \sin t = \frac{1}{70} \int 35 \cos t \; d(140t) \qquad (2)$$

$$50a_1 = \frac{1}{70} \int_{t=0}^{t=2\pi} 50f(t) \; d(70 \sin t) \qquad (3)$$

$$50a_1 = -\frac{1}{70} \int_{t=0}^{t=2\pi} 50f(t) \; d(70 \cos t) \qquad (4)$$

$$50a_2 = \frac{1}{70} \int_{t=0}^{t=4\pi} 50f(t/2) \; d(35 \sin t) \qquad (5)$$

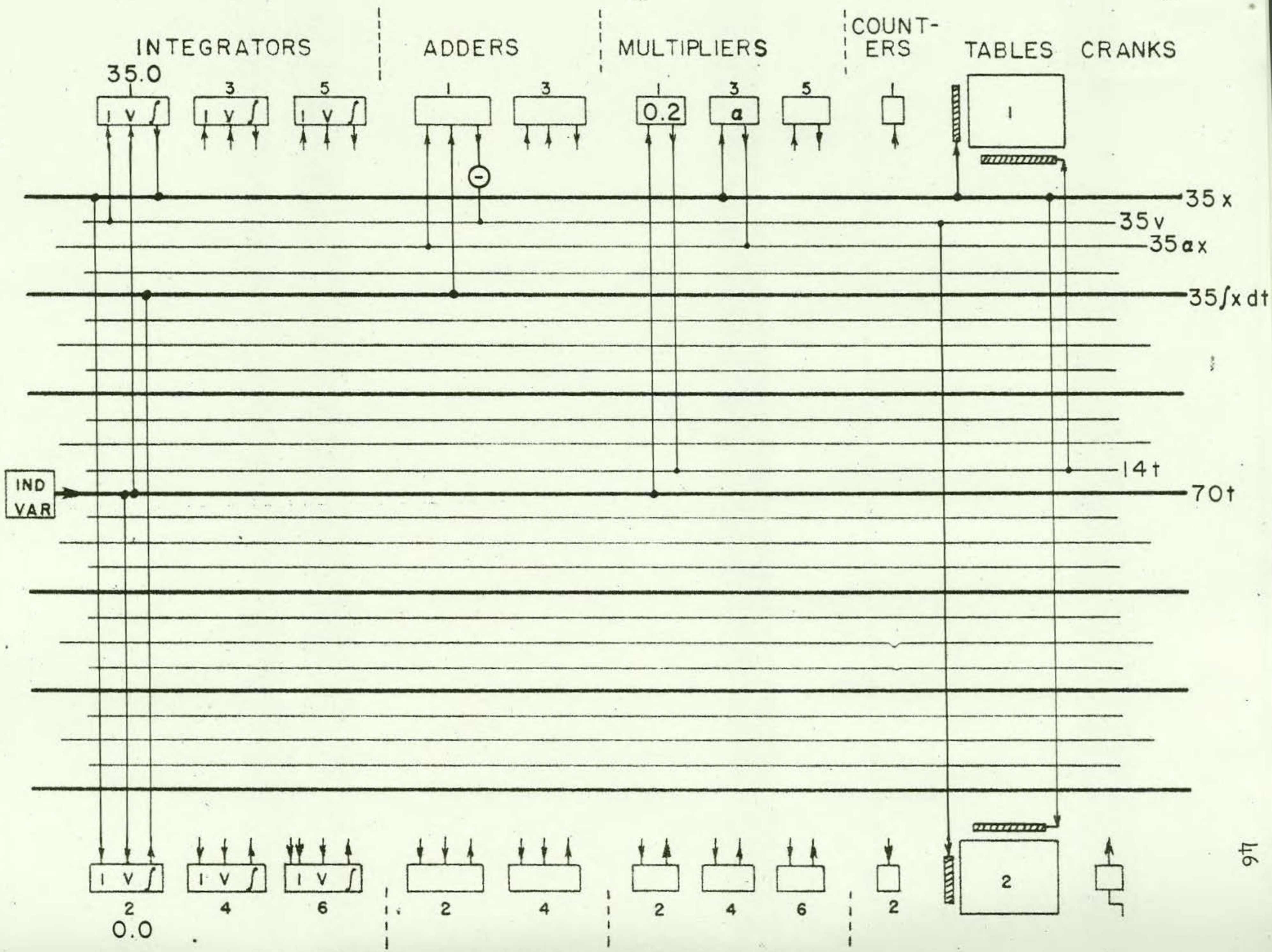
$$50b_2 = -\frac{1}{70} \int_{t=0}^{t=4\pi} 50f(t/2) \; d(35 \cos t) \qquad (6)$$

For the connection chart, see page 51.

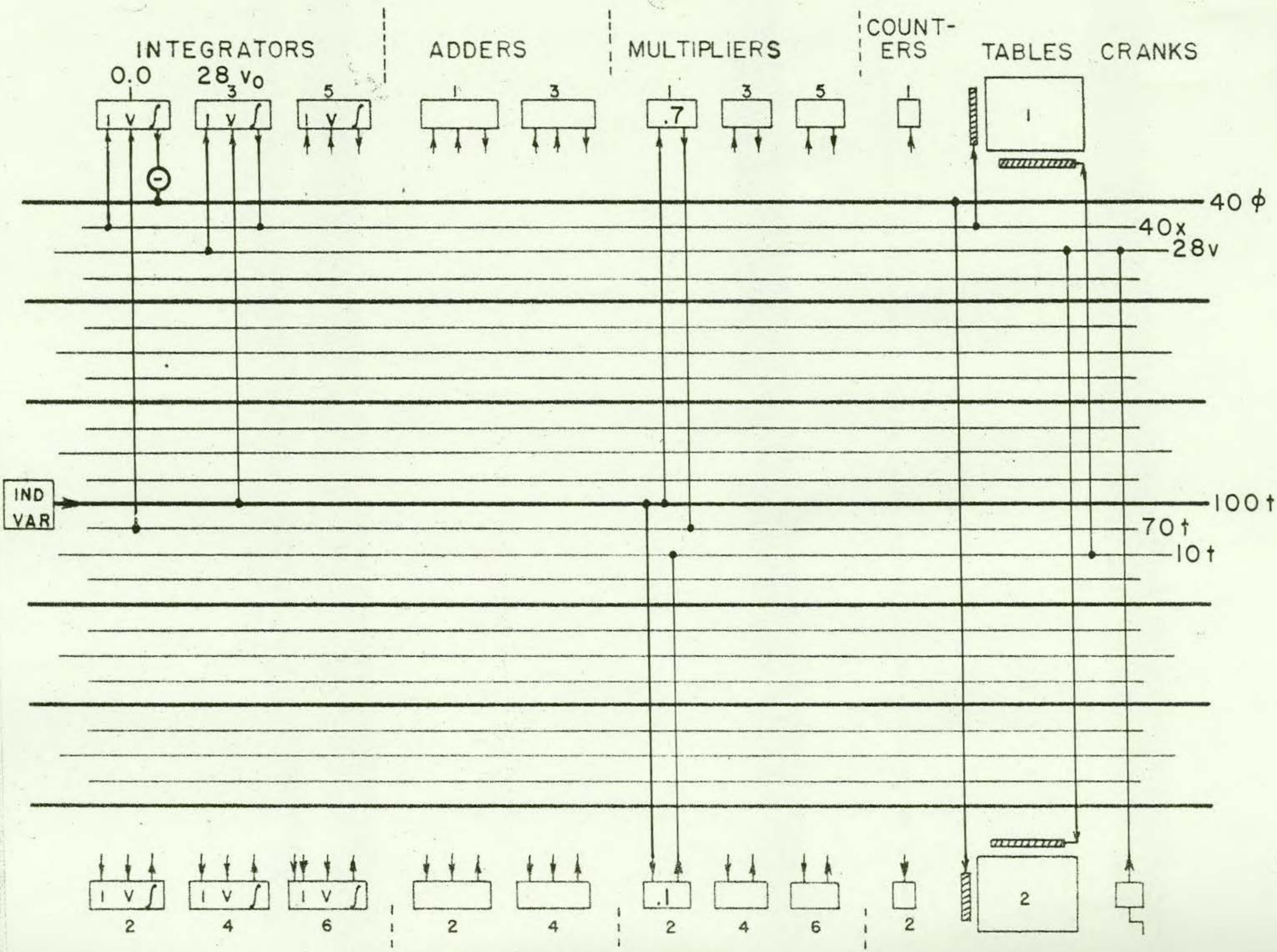
7. Mathieu's equation:

$$\frac{d^2x}{dt^2} + (a + b \cos \omega\, t)\, x = 0$$

Planning the setup for this is presented to the reader as an exercise.



DIFFERENTIAL ANALYSER CONNECTION CHART



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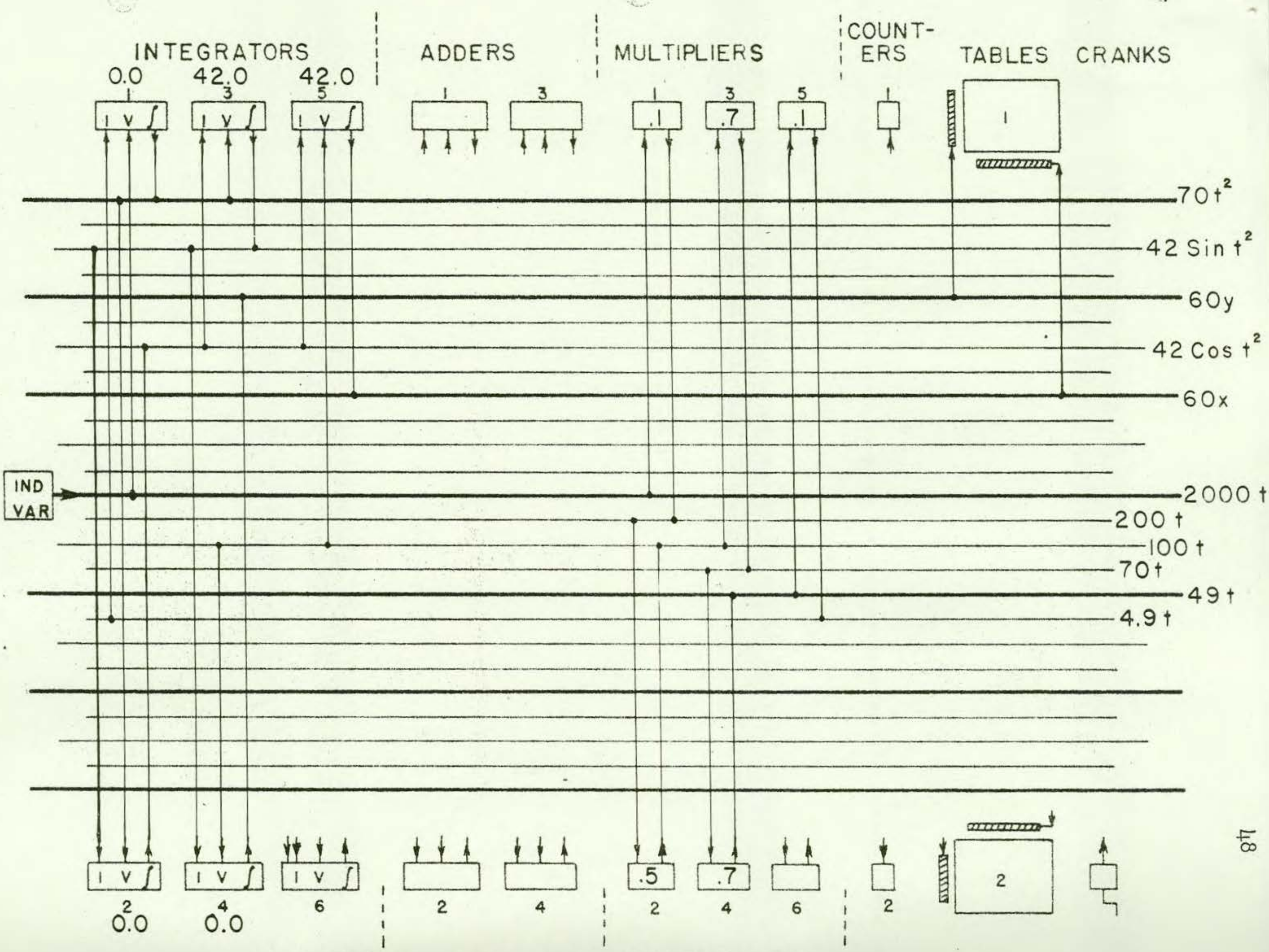

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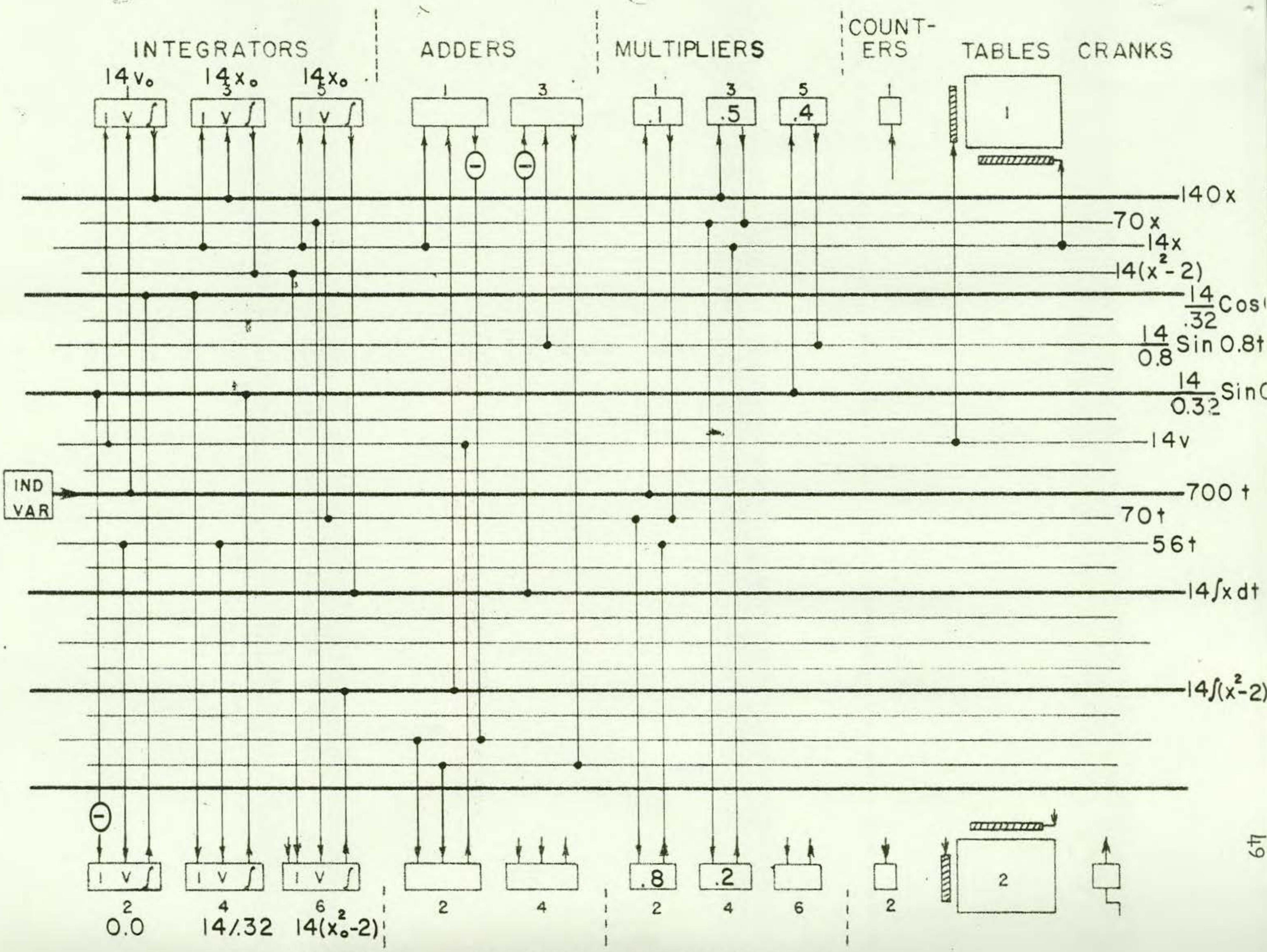

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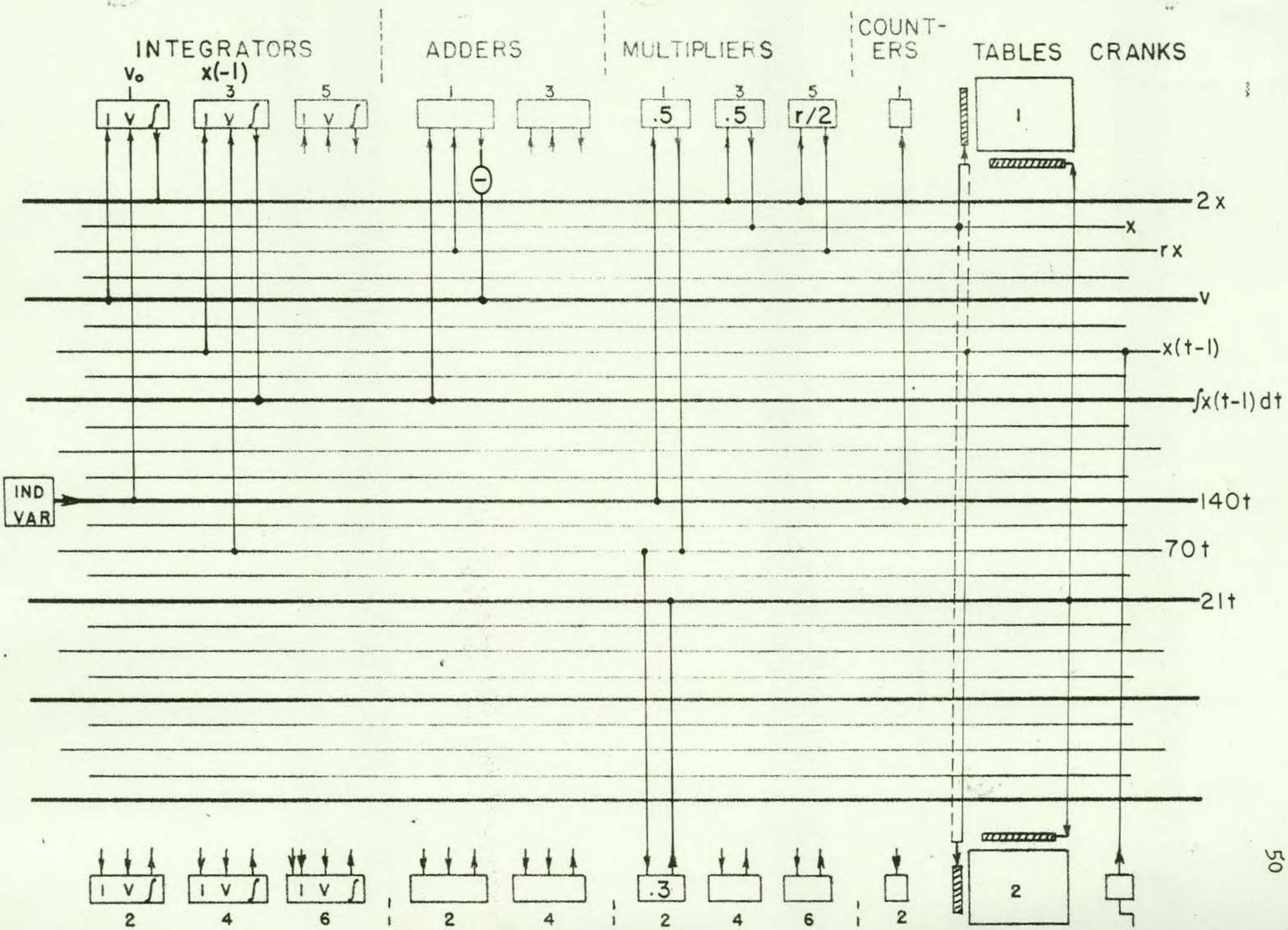

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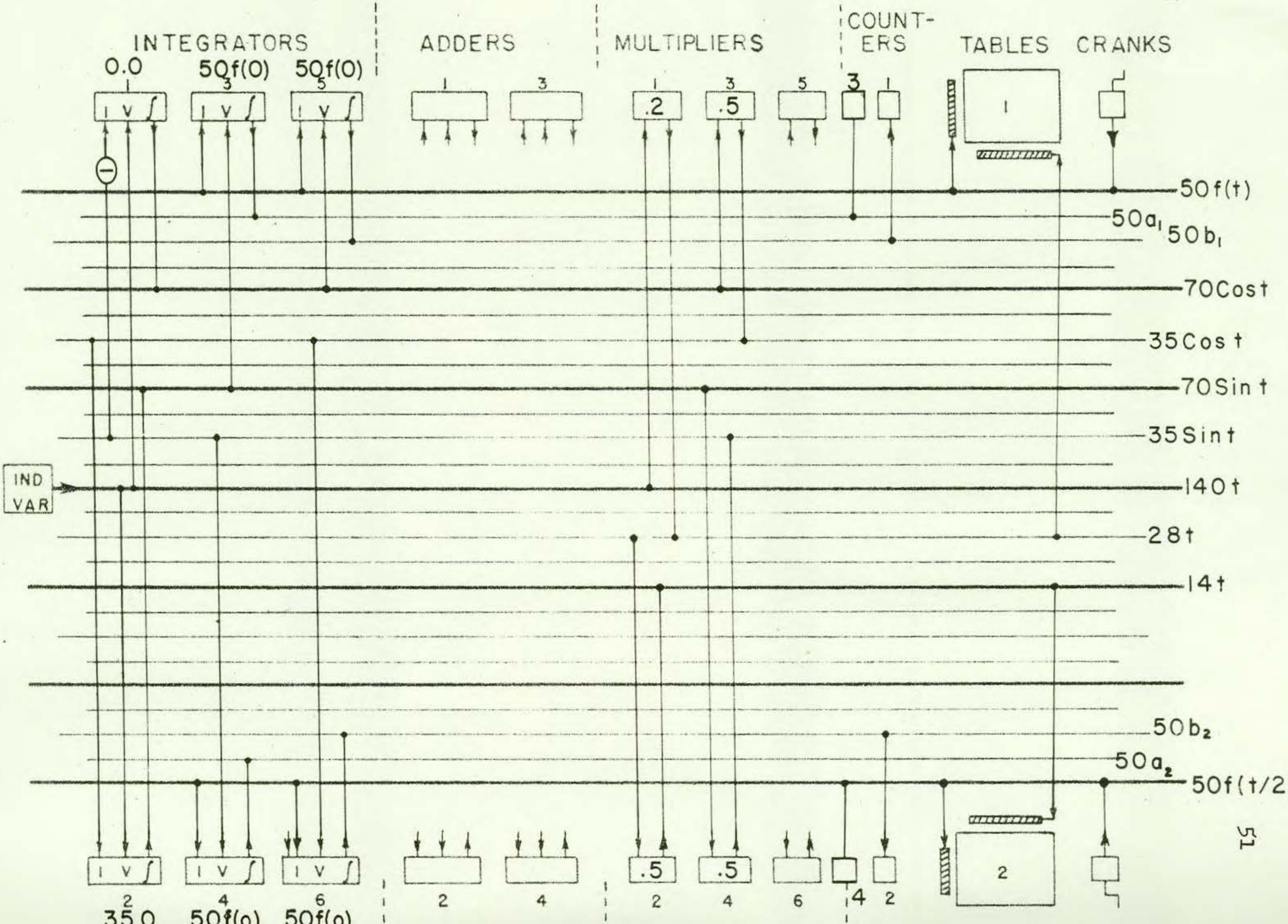

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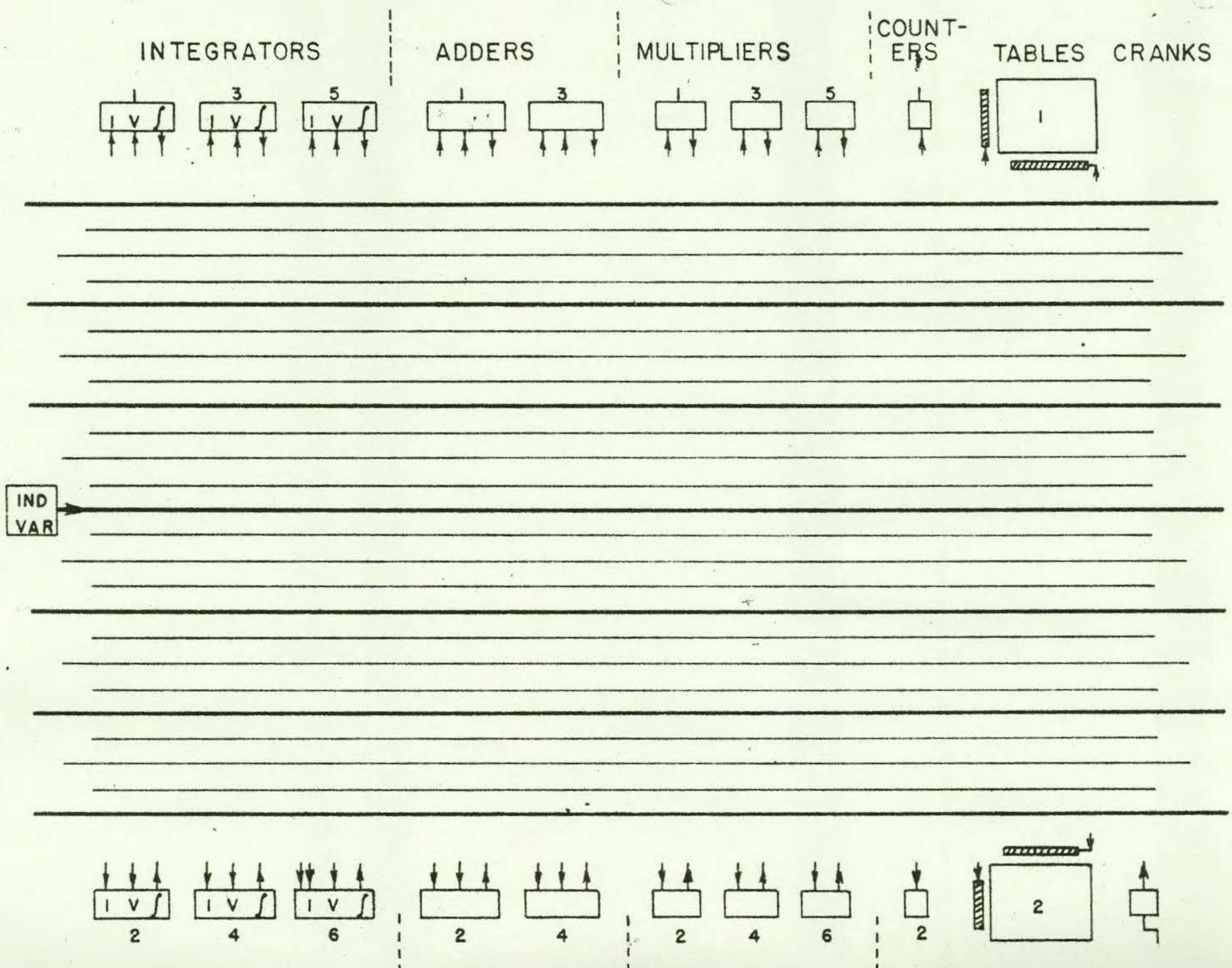

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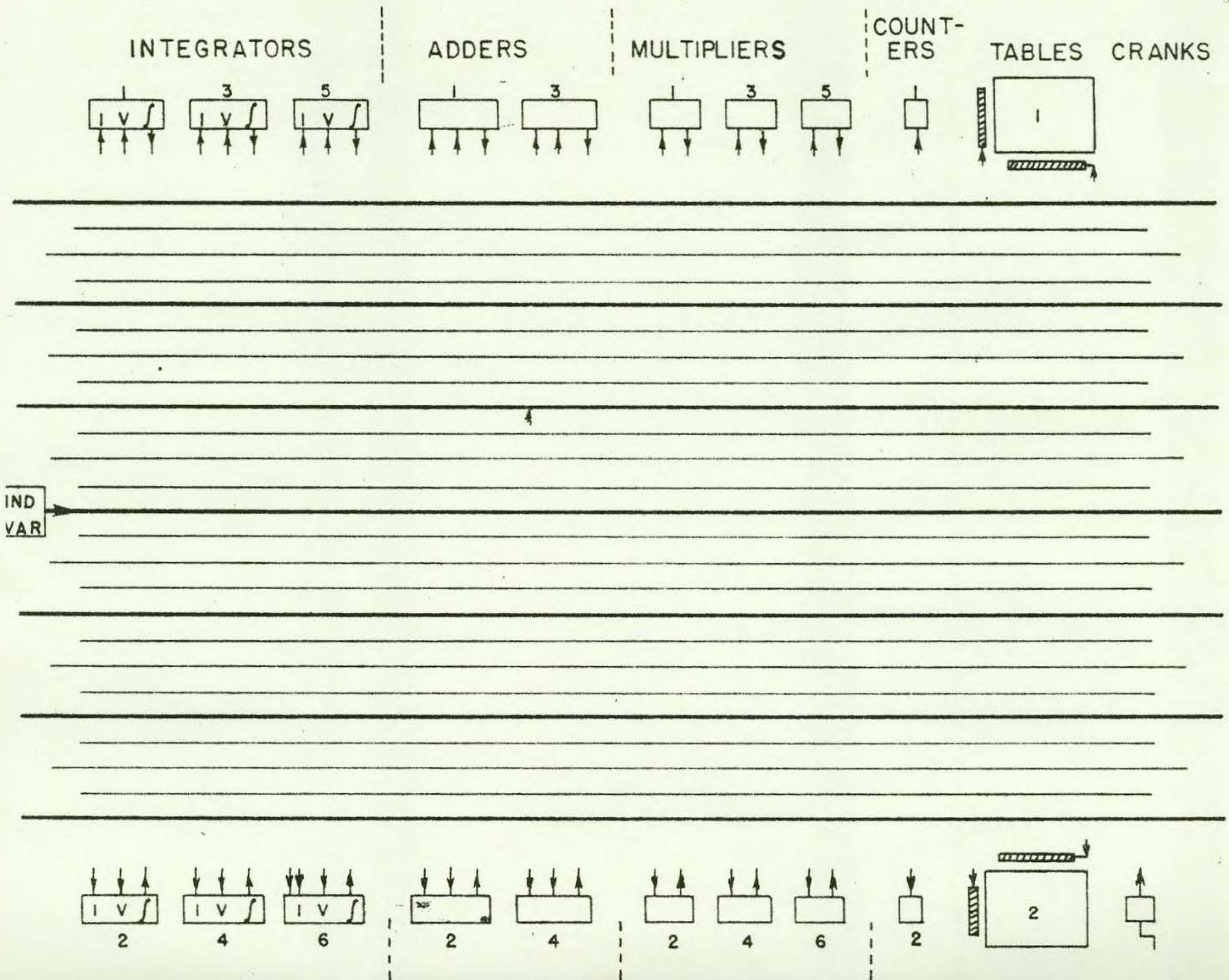

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