CHAPTER 3

Switching Networks

Before proceeding to the arithmetic operations specifically, this chapter will be devoted to switching networks in general. As will be illustrated in subsequent chapters, arithmetic operations are performed in calculators largely through the assembling of the calculator components to form switching networks of various descriptions. Also, switching networks have application in parts of a calculator other than the part which does the actual adding, multiplying and other operations. In particular, the control portion of a calculator is comprised almost entirely of switching networks. Further, switching networks are of interest in many machines and devices such as elevator controls, telephone switchboards, code and cipher machines, and railway signalling systems, which are not calculators in the usual sense of the word at all.

In broad terms a switching network is any digital device to which input signals may be applied and from which output signals may be obtained that are some prescribed function of the input signals. In the examples to be described it will be assumed that the signals are all two-valued; that is, on a given signal line, a signal will either be present or will not be present. In other words, a signal can be considered as having the value, 1 or 0, according to whether it is in existence or not. There is no inherent reason why multi-valued signals could not be used, but very little practical use has been made of them because of the difficulties in designing suitable physical components. Reference will be made both to steady-state and pulse-type signals. In most cases either type of signals may be assumed; those instances which require one kind of signal or the other will be apparent from the text from the nature of the switching network.

Some of the most elementary networks were discussed in the previous chapter. More complicated, in fact even very complicated, switching networks usually may be quite easily assembled in a straightforward manner. The difficult part of the task is finding an arrangement which has the minimum or a reasonable number of components or which meets some other requirement such as speed of operation. Frequently, further complications are introduced which might not be apparent from the functional block diagrams containing only the switching elements. One such complication is encountered when diodes are used. When a signal passes through a succession of alternate "and" and "or" switches its amplitude rapidly diminishes. If the circuits are properly designed, simple cathode follower circuits may be used at intermediate points to produce the required current gain. With a large number of stages, voltage amplification as well as current amplification will be required. Of course, the signal amplification equipment must be considered as well as switching elements themselves.

Even with components which require no separate amplifiers two or more different types of components may be used, and then it is necessary to find the lowest possible "weighted count" in the various possible switching configurations because the different types may have variations in cost or desirability.

In spite of the virtual impossibility of finding a general solution to network problems which must include engineering considerations, a few aids and tricks are known. Some of the more useful procedures for finding suitable switching networks will be pointed out.

Elemental Form of Network. Any switching function involving a single output signal which is a function of a set of simultaneously applied input signals may be reduced to an "elemental form". Here, the term, elemental form, means a Boolean algebra expression (or the equivalent physical circuit) where the desired result is obtained by a set of "and" terms combined by an "or" relationship, where each "and" term contains all of the variables. It is possible to find this form in any given instance merely by noting those combinations of input signals for which an output signal is desired.

For example, with three input signals, A, B, and C, there are eight and only eight possible input combinations. Each input combination may be represented by an "and" term such as $\bar{A}B\bar{C}$, which has the meaning that a signal is applied to input B but not to inputs A or C. Any given switching function may be specified by a listing of the combinations of input signals which will produce an output signal. Since the listing implies an "or" relationship, it follows that any switching function may be represented by an expression of the form,

$$\bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C} + \cdots,$$

where only those terms which are to yield an output signal are included. Of course, when a switching function is encountered in a practical problem, it may not appear in this elemental form, but through Boolean algebra manipulations it is not difficult to alter the representation to fit this form. For instance, the elemental form of the function $\overline{(AB + \bar{\bar{C}})}$ may be found by proceeding through the following steps.

$$\overline{(AB + \bar{\bar{C}})} = \overline{AB}\,C$$

$$= (\bar{A} + \bar{B})\,C$$

$$= \bar{A}BC + A\bar{B}C + \bar{A}\bar{B}C$$

Through a study of the elemental forms of switching networks the total number of different switching functions may be determined. In the case of only one input variable, the networks are all trivial; nevertheless, there are four of them as represented by the expressions, 0, A, $\bar{A}$ and $A + \bar{A}$. The first is an open circuit, the second is a straight connection between the input and output, the third is an inverter and the last is a steady output signal independent of the input. With two input variables there are sixteen different switching functions as follows:

0	$\bar{A}\bar{B} + \bar{A}B + A\bar{B}$
$\bar{A}\bar{B}$	$\bar{A}\bar{B} + \bar{A}B + AB$
$\bar{A}B$	$\bar{A}\bar{B} + A\bar{B} + AB$
$A\bar{B}$	$\bar{A}B + A\bar{B} + AB$
AB	$\bar{A}\bar{B} + \bar{A}B + A\bar{B} + AB$
$\bar{A}\bar{B} + \bar{A}B$	
$\bar{A}\bar{B} + A\bar{B}$	
$\bar{A}\bar{B} + AB$	
$\bar{A}B + A\bar{B}$	
$\bar{A}B + AB$	
$A\bar{B} + AB$	

As will be explained shortly, most of these expressions can be simplified considerably. With three input variables a total of 256 different switching functions are possible although many of them are merely rearrangements of the variables.

In general, the number of switching functions may be found by noting that the total number of different input combinations is equal to 2^N, where N is the number of input variables. It may be desired to have an output signal for any set of input combinations. Therefore, each of the sets can be represented by a binary number of 2^N digits with a 1 or a 0 in the number meaning that the corresponding input combination causes or does not cause, respectively, an output signal. Since there is a one-to-one correspondence between the sets and the binary numbers, the total number of switching combinations is equal to 2 raised to the 2^Nth power.

Simplifying the Elemental Form. The elemental form of representation of a switching function does not, of course, necessarily yield a representation of the physical switching circuit which is most economical in terms of the number of components required. In fact, in the wide majority of instances, the circuit may be simplified. In this section only those circuit configurations which involve a set of "and" switches combined by an "or" switch will be considered.

When hunting for unnecessary components to remove from an "and-to-or" switching circuit (such as a circuit in its elemental form) a good way to start is to look for combinations of variables in any of the forms, $X\overline{Y} + XY$, $X + XY$, or $\overline{X} + XY$ because each of these expressions can be reduced as follows:

$$X\overline{Y} + XY = X(\overline{Y} + Y) = X$$

$$X + XY \quad = X(1 + Y) = X$$

$$X + \overline{X}Y \quad = (X + \overline{X})\,(X + Y) = X + Y.$$

As an illustration of the use of these relationships, consider the switching function, $\overline{A}\,\overline{B} + \overline{A}B + A\overline{B}$, which is found in the list of the sixteen two-input functions. The first two of the three terms in the expression are of the type represented by the first equation above, and the function may therefore be reduced to $\overline{A} + A\overline{B}$, which in turn may be reduced to $\overline{A} + \overline{B}$ as determined by the third equation. Through the use of the three equations it may be found quickly that the entire list of sixteen two-input switching functions may be reduced to the following list of relatively simple expressions.

0	
$\overline{A}\,\overline{B}$	$\overline{A} + \overline{B}$
$\overline{A}B$	$\overline{A} + B$
$A\overline{B}$	$A + \overline{B}$
AB	$A + B$
$\overline{A}$	
$\overline{B}$	
$\overline{A}\,\overline{B} + AB$	
$\overline{A}B + A\overline{B}$	
B	
A	

When working with switching functions involving three or more input variables the three equations presented in the previous paragraph can be represented in a more general manner.

$$f(X_n)\overline{g(Y_m)} + f(X_n)g(Y_m) = f(X_n)$$

$$f(X_n) + f(X_n)g(Y_m) = f(X_n)$$

$$f(X_n) + \overline{f(X_n)}g(Y_m) = f(X_n) + g(Y_m)$$

Here, X_n signifies a set, X, of n variables, and Y_m signifies a set, Y, of m variables. It is not a requirement that any given variable appear in only one set; it may appear in both sets. A notation such as $f(X_n)$ means any Boolean algebra function of X_n. In this section most of the functions are limited to simple "and" combinations of the variables, but it should be understood that this limitation is not general.

As another example of the procedure, consider the function,

$$ABC + A\overline{B}C + AB\overline{C}D.$$

If the first two terms of the expression are examined, it will be observed that AC is common to both of them and may be taken as a single variable or function. Therefore, the expression reduces to $AC + AB\overline{C}D$. Further simplification cannot be achieved through direct application of any of the three equations, but if the common variable, A, is factored to yield $A(C + B\overline{C}D)$, then BD may be taken as a single variable. The resulting expression is $A(C + BD) = AC + ABD$. In this, as in most other examples, several different sequences of steps could have been used to achieve the same result.

Sometimes when simplifying a switching function it is easier, at least from the standpoint of visualization, to use a given term more than once in the process. For example:

$$\overline{A}B\overline{C} + A\overline{B}\overline{C} + AB\overline{C} = \overline{A}B\overline{C} + A\overline{B}\overline{C} + AB\overline{C} + AB\overline{C}.$$

The third term in the left-hand side of the equation is recorded twice in the right-hand side (possible because $X = X + X$). Then, by considering the first and third terms together and the second and fourth terms together it is found that the expression is equivalent to $B\overline{C} + A\overline{C}$.

A more striking example of the power of the three generalized simplifying equations is illustrated in the following example.

$$\begin{aligned}(AB + C) + \overline{(AB + C)}(CD + A) &= (AB + C) + (CD + A)\\ &= A + C\end{aligned}$$

The simplified form is found very quickly by considering the expressions within the brackets as individual variables. Without the generalized equations, a process somewhat as follows would be necessary.

$$(AB + C) + \overline{(AB + C)}(CD + A) = (AB + C) + \overline{AB}\,\overline{C}\,(CD + A)$$

$$= (AB + C) + (\overline{A} + \overline{B})\,\overline{C}\,(CD + A)$$

$$= (AB + C) + A\overline{B}\,\overline{C}$$

$$= A(B + \overline{B}\,\overline{C}) + C$$

$$= A(B + \overline{C}) + C$$

$$= AB + A\overline{C} + C$$

$$= AB + A + C$$

$$= A + C$$

<u>A Difficulty and Its Solution.</u> Occasionally switching functions of the simple "and-to-or" variety contain superfluous terms which cannot be detected by the methods described in the previous section. An example is $AB + \overline{B}\,\overline{C} + A\overline{C}$, where the third term, $A\overline{C}$, is superfluous and may be dropped without altering the value of the function.

One way to show that the term is superfluous is through multiplying it by $B + \overline{B}$, which is equal to 1 and therefore does not change its value.

$$AB + \overline{B}\,\overline{C} + A\overline{C} = AB + \overline{B}\,\overline{C} + A\overline{C}(B + \overline{B})$$

$$= AB + \overline{B}\,\overline{C} + AB\overline{C} + A\overline{B}\,\overline{C}$$

$$= AB(1 + \overline{C}) + (1 + A)\overline{B}\,\overline{C}$$

$$= AB + \overline{B}\,\overline{C}$$

While the steps illustrated here may seem perfectly straightforward, it is sometimes quite puzzling to find the proper steps when an example of this type is encountered for the first time. Experience and practice is a great help in knowing which steps of the multitude that are possible are most likely to be fruitful in finding simplifications in the switching function. A study of the following additional three examples will aid in discerning patterns that are likely to signify superfluous terms.

$$A\overline{B} + BC + AC = A\overline{B} + BC$$
$$\overline{A}\,\overline{B} + B\overline{C} + \overline{A}\,\overline{C} = \overline{A}\,\overline{B} + B\overline{C}$$
$$\overline{A}B + \overline{B}C + \overline{A}C = \overline{A}B + \overline{B}C$$

By using a "testing" process superfluous terms can be detected in a positive manner. To test a given term, the values of the input variables are observed which cause an output signal because of the term being tested. These values of the variables are then inserted in all of the other terms, and if it is found that an output signal is always created through at least one other term, it is known that the term being tested is superfluous. The process will be illustrated by testing the term $A\overline{C}$ in the previous example of $AB + \overline{B}\overline{C} + A\overline{C}$. For $A\overline{C}$ to cause an output signal (that is, to cause the value of the switching function to be 1) it must be that $A = 1$ and $C = 0$. But in this case the first two terms yield $(1)B + \overline{B}(1)$, which is always equal to 1. Therefore, the $A\overline{C}$ is superfluous. By applying this test to all terms in an "and-to-or" switching function, it is possible to find with certainty any that may be superfluous.

Another illustrative example is the switching function,

$$A\overline{B} + B\overline{C} + C\overline{A} + \overline{A}B + \overline{B}C + \overline{C}A.$$

A test on any one of the six terms will indicate that it is superfluous. The result of testing the first term (set $A = 1$ and $B = 0$) is

$$0 + 0 + 0 + (1)\,C + \overline{C}(1) = 1,$$

and analogous results are obtained from tests of any one of the other terms. However, it should not be concluded that all terms are superfluous and that the function is equivalent to 0. After testing one term and finding that it is superfluous, it should be eliminated before testing another. In particular after eliminating the first term, the second or third terms will be found superfluous, but the last three will not be superfluous. It happens that in this example the switching function may be simplified to any one of two equivalent functions which are:

$$A\overline{B} + B\overline{C} + C\overline{A} = \overline{A}B + \overline{B}C + \overline{C}A.$$

Although individual variables which are superfluous can usually be found by methods described previously, the testing procedure may be used for them also. A slight modification in the procedure is required. For example, in the switching function, $AB + A\overline{B}C$ the appearance of $\overline{B}$ in the second term is superfluous. To test this particular appearance of the variable it must be observed that it will contribute to the output only when $A = 1$ and $C = 1$. But then the expression is equivalent to $(1)(B) + (1)\overline{B}(1) = 1$. Since the value of the expression is always 1 when $A = 1$ and $C = 1$, the $\overline{B}$ factor in the second term may be eliminated.

As another example consider the switching function, $AB + A\overline{B}C + \overline{A}\overline{B}$. The appearance of A in the second term or the appearance of $\overline{B}$ in the second term, but not both, is superfluous as may be determined by testing. The test for A in the second term is to note that $B = 0$ and $C = 1$ when this term contributes to the output. Since in this case the switching function is $0 + A(1)(1) + \overline{A}(1)$, which is always 1, it follows that the A in the second term is not needed. The same result could have been found with previously described methods by factoring $\bar{B}$ out of the second and third terms as a first step. By similar procedures it may be shown that $\bar{B}$ in the second term could be eliminated instead of A.

Another Difficulty and Its Solution. In all of the examples described previously the simplifications were accomplished through eliminating unnecessary variables or terms in the expression. Occasionally, when applying the methods which have been described it is possible to proceed into a "trap". The trap is a situation where an expression is found which contains no superfluous variables or terms, as can be proven by testing, but yet is not the simplest "and-to-or" expression which represents the desired switching function.

An example is the following switching function, which when represented in its elementary form is

$$A\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}\bar{B}C + \bar{A}BC + A\bar{B}C + AB\bar{C}.$$

By grouping first and fifth, the second and fourth, the first and sixth, and the third and fourth terms in pairs the function may be simplified to:

$$A\bar{B} + \bar{A}B + A\bar{C} + \bar{A}C,$$

which contains no superfluous variables or terms. Yet, it is possible to form the function with less terms than this. If the terms in the original expression had been grouped with the first and fifth, the second and sixth, and the third and fourth terms in pairs, the result would have been

$$A\bar{B} + B\bar{C} + C\bar{A}.$$

Although a criterion for simplicity is sometimes difficult to define, by almost any conceivable standards, this expression is simpler than the previous one. It may be recognized that this example is substantially the same as a previous one; by a different grouping of the original terms the different, but equivalent expression,

$$\bar{A}B + \bar{B}C + \bar{C}A,$$

could have been obtained.

What is needed in avoiding traps of the type just described is a systematic way of finding all possible combinations of simplified forms so that the most desirable one may be selected. One such system involves expanding the function to its elemental form (if it is not already in its elemental form). Then, through repeated applications of the formula, $X\bar{Y} + XY = X$, in a systematic manner all "basic" terms in the expression may be found, where a basic term is any correct term which contains no superfluous variables. In the previous example a total of six basic terms ($A\bar{B}$, etc.) are found by pairing the terms of the elemental form in various ways. Actually, it is not known that these terms are basic until each of them has been compared with all others and it has been determined that further reductions cannot be made. After finding all basic terms a table is made that indicates which of them are contained in each term in the elemental form. For the example cited the table is shown in Table III-1, where an x indicates that the basic term listed in the corresponding row is contained in the elemental term at the top of the corresponding column.

	$A\bar{B}\bar{C}$	$\bar{A}B\bar{C}$	$\bar{A}\bar{B}C$	$\bar{A}BC$	$A\bar{B}C$	$AB\bar{C}$
$A\bar{B}$	X				X	
$\bar{A}B$		X		X		
$B\bar{C}$		X				X
$\bar{B}C$			X		X	
$C\bar{A}$			X	X		
$\bar{C}A$	X					X

Table III-1. Table for simplifying $A\bar{B} + \bar{A}B + A\bar{C} + \bar{A}C$.

Through inspection of the table and a systematic selection of the basic terms, all possible combinations of basic terms which will be equivalent to the original expression can be found. The basic terms must, of course, be selected such that each term in the elemental form is represented at least once (that is, such that the selected terms will represent at least one x in each column).

As another example, it may be shown by similar procedures that the following four-term expression is a trap in that it contains no superfluous terms or variables and that it may be reduced to either of two different three-term expressions.

$$ACD + A\bar{B}C + \bar{A}BC + \bar{A}C\bar{D} = ACD + \bar{A}BC + \bar{B}C\bar{D}$$

$$= A\bar{B}C + \bar{A}C\bar{D} + BCD$$

The systematic procedure for analyzing switching functions is sometimes useful even when no simplifications can be achieved. As an example of this case consider the switching function, $ABC + A\bar{B}\bar{D} + B\bar{D}$. When this expression is expanded to its elemental form by multiplying individual terms by expressions of the form $X + \bar{X}$, it is found that the elemental terms shown at the top of Table III-2 are present after duplicates have been eliminated

	$ABCD$	$ABC\bar{D}$	$A\bar{B}CD$	$AB\bar{C}\bar{D}$	$\bar{A}BC\bar{D}$	$\bar{A}B\bar{C}\bar{D}$	$A\bar{B}\bar{C}D$
ABC	X	X					
ACD	X		X				
$B\bar{D}$		X		X	X	X	
$A\bar{B}D$			X				X

Table III-2. Analysis of $ABC + A\bar{B}D + B\bar{D}$.

When the terms of the elemental form are grouped by twos in all possible combinations a set of three-variable terms is found as follows: ABC, ACD, $AB\bar{D}$, $BC\bar{D}$, $A\bar{B}D$, $B\bar{C}\bar{D}$, and $\bar{A}B\bar{D}$. Of these the third and seventh and also the fourth and sixth may be paired with each pair yielding the term, $B\bar{D}$. The basic terms of the expression are, therefore, ABC, ACD, $B\bar{D}$, and $A\bar{B}D$. It happens that in this example three of the basic terms were in the original expression, but one new one has been found. From Table III-2 it is easily determined that $B\bar{D}$ is a necessary basic term because it is the only one appearing in the columns for $AB\bar{C}\bar{D}$ and two other elemental terms. For a similar reason, $A\bar{B}D$ is necessary. Since both of these terms must appear in any simplified form of the switching expression, the only elemental term not accounted for is $ABCD$. Either ABC or ACD may be chosen, and the other one becomes superfluous. Although no reductions in the number of terms or variables has been achieved, it has been found that $ACD + B\bar{D} + A\bar{B}D$ is an alternate expression; this expression would have been extremely difficult to find in any haphazard way. One possible value of the alternate expression could arise from physical characteristics of the circuits supplying the input signals. If it happened that the circuit supplying B were capable of operating only one "and" switch while the circuit supplying D could operate two, the alternate would be preferable.

Separation of Variables. If it is possible to group the terms in an expression so that none of the variables appearing in any one group appears in any other group, then the network simplifying procedures which have been described can be applied to the individual groups instead of the entire expression with a considerable saving in effort. Consider the switching expression,

$$A\bar{B} + BC + AC\bar{D} + D + A\bar{B}E + \bar{E}.$$

To analyze this expression thoroughly by expanding it to its elemental form would be laborious, but by procedures described previously it may be quickly simplified to:

$$(A\bar{B} + BC + AC + A\bar{B}) + (D + \bar{E}) = A\bar{B} + BC + D + E.$$

Note that the terms in the first set of parenthesis contain only A, B, and C, and the terms in the second set contain only D and E. Although a formal proof is somewhat involved, it can be shown that the expression obtained from simplifying the groups separately is the same as would be obtained by handling all of the original terms together.

Factoring. When it is not a requirement that the switching function remain of the "and-to-or" type, a reduction in the number of components can be achieved frequently through the use of simple factoring. An example is $AB + AC$ where the A may be factored out to yield $A(B + C)$. If diode switching is employed where the number of diodes in the circuit is equal to the number of input signals to each "and" and "or" switch, a total of six diodes is required for a circuit conforming to the original expression, but only four are required for the factored form.

Factoring does not always yield a simplification; in fact, in some instances more components are required and other disadvantages are introduced. Consider the switching function, $ABC + ADE + F$, which requires nine diodes. If the A is factored to yield $A(BC + DE) + F$, a total of ten diodes becomes necessary. Furthermore, some of the input signals, B for instance, must proceed through an "and-to-or-to-and-to-or" switching sequence. Multiple level switching such as this is accomplished only with difficulty when diodes or some other types of switching components are used.

Converting to an "Or-to-and" Type of Circuit. Through repeated applications of the formula, $X + YZ = (X + Y)(X + Z)$, any switching function in "and-to-or" form may be converted to "or-to-and" form. In the previous chapter this procedure was used to develop the following equality:

$$AB + CD = (A + C)(A + D)(B + C)(B + D)$$

In this case, no simplification is obtained; instead, the expression becomes more complex. However, if the original expression had been $AC + AD + BC + BD$, it could be shown by the same procedure (or by simple factoring in this case) that it is equivalent to $(A + B)(C + D)$, which corresponds to a substantially simpler circuit.

From the above example it might be expected that the probability of finding a more complex or a simpler circuit is the same. For random switching functions this situation is true because there is a one-to-one correspondence between the set of all possible "and-to-or" circuits and all possible "or-to-and" circuits, which of course includes all possible circuits. For the switching functions encountered in practical applications there is considerable question about their randomness. Although no conclusive data is known, some rough surveys have indicated that the "and-to-or" type which becomes more complex upon conversion is the more prevalent.

Factors other than the number of components frequently contribute to the "simplicity" of a circuit. One such factor is standardization. It is sometimes desirable for reasons of standardization to use only "and-to-or" circuits or only "or-to-and" circuits. In other cases, it may be that switching components are chosen which function much more satisfactorily in one type of circuit than the other. Since conversion is possible, either type may be used regardless of the nature of the switching functions. If it happens that the type most suited to the components requires a larger number of components than the other type, this disadvantage may be avoided in some machines by redefining the representation of a "signal" and "no signal". For example, if a signal is normally represented by a relatively positive voltage, changing to a relatively negative voltage for a signal representation will cause all "and" and "or" functions to be interchanged.

The significance of the "or-to-and" form of switching networks is of interest from the standpoint of mental visualization. In some cases the "or-to-and" form is a distinct aid in visualizing the true nature of a switching function, but in most cases it seems to be only an artificial sort of representation. In particular, the elemental "and-to-or" form specifies in an easily visualized manner those combinations of input signals which create an output signal. Elemental "or-to-and" functions can be worked out, but their usefulness, if any, is difficult to imagine. For example, the elemental "or-to-and" form for $(A + B)(\overline{B} + C)$ would be

$$(A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(\overline{A} + \overline{B} + C),$$

which seems to obscure the true nature of the function.

One example where the "or-to-and" form might be useful for visualization purposes is

$$A\overline{B} + B\overline{C} + C\overline{A} = (A + B)(A + \overline{C})(\overline{B} + \overline{C}) + C\overline{A}$$

$$= (A + B + C)(\overline{A} + \overline{B} + \overline{C}).$$

From the "or-to-and" form it may be observed readily that the function causes an output signal to be generated in any situation where at least one of the three input variables, A, B, and C, is 1 and at the same time at least one of the variables is 0. This visualization of the function is not so clearly observable from the "and-to-or" form.

Two reasons for the selection of the particular Boolean algebra notation which is being used here can now be explained. One reason is related to the fact that the majority of switching functions encountered in practical applications yield a less complex switching circuit when in the "and-to-or" form. The other pertains to the fact that the "and-to-or" form is usually more useful for mental visualization of the function. For both reasons the use of + signs and parenthesis is the lesser when the "and" and "or" function notations are made to correspond to sums and products, respectively. In applications where the "or-to-and" form is known to predominate, the opposite convention may be used in order to decrease the incidence of + signs and parentheses.

A peculiar property of the conversion process is that, if the factors in the "or-to-and" expression which is obtained after a conversion are arbitrarily altered to form terms in a new "and-to-or" expression, the new expression will frequently contain superfluous terms which cannot be removed by the more elementary procedures. For example, the expression, $AB + \bar{B}D + C\bar{D}$, when converted to "or-to-and" form is equal to

$$(A + \bar{B} + C)(A + \bar{B} + \bar{D})(A + C + D)(B + C + D).$$

When the variables in the factors are arbitrarily grouped to generate a new "and-to-or" expression, the following switching function is obtained:

$$A\bar{B}C + A\bar{B}\bar{D} + ACD + BCD.$$

Either the first or the third term, but not both, is superfluous as may be proven by testing or by analyzing the expression more completely through use of methods which have been described. By inverting the new expression, it is found to be equal to

$$(A + B)(A + C)(A + D)(\bar{B} + C)(\bar{B} + D)(C + \bar{D}),$$

which may be arbitrarily altered to

$$AB + AC + AD + \bar{B}C + \bar{B}D + C\bar{D}.$$

It can be shown that the second, third, and fourth terms are all superfluous in this "and-to-or" expression. Note that it is equivalent to the expression used at the start of the example. It is always true that after two conversions with arbitrary alterations of this type that the result will be equivalent to the original expression.

The conversion property described in the previous paragraph has academic usefulness in that it may be used to find new examples and problems for students. Also, incidentally, through mere rearrangement of terms and inversions of variables, many of the examples given in this chapter can be rendered substantially unrecognizable and can then be used as problems. Three miscellaneous examples which provide good practice in Boolean algebra manipulations and which were originally discovered through selecting random functions and running them through two successive conversions are listed below.

$$A\bar{B} + AC + A\bar{D} + \bar{B}D = ABC + \bar{B}D + A\bar{D}$$

$$= AC + A\bar{D} + \bar{B}D$$

$$ABC + A\bar{B}D + ACD + \bar{A}\bar{C} + \bar{B}\bar{C}D = ABC + ACD + \bar{A}\bar{C} + \bar{B}\bar{C}D$$

$$= ABC + A\bar{B}D + \bar{A}\bar{C}$$

$$ABC + ACD + \bar{A}\bar{B}C + \bar{A}C\bar{D} + BC\bar{D} + \bar{B}D = ACD + \bar{A}C\bar{D} + BC\bar{D} + \bar{B}D$$

$$= ACD + \bar{A}\bar{B}C + BC\bar{D} + \bar{B}D$$

$$= ABC + \bar{A}\bar{B}C + BC\bar{D} + \bar{B}D$$

$$= ABC + \bar{A}C\bar{D} + \bar{B}D$$

When performing conversions, if relationships of the type,

$$(W + X + Y + Z)(W + X) = W + X;$$

are kept well in mind and used where possible, much work can be avoided by simplifying the expressions obtained at intermediate steps in the process. The correctness of this relationship can be established readily through methods described previously.

Situations Where Some Combinations of Input Variables Will Not Exist. In all previous examples it was assumed that input signals might be applied in any possible combination. In many applications encountered in calculators and other machines which use switching networks, it may be discovered upon detailed examination of the problem that certain combinations of input signals will never exist. When this situation is found, it is frequently possible to find a circuit which uses less components or is otherwise simpler than one which must respond properly to all possible combinations of input signals.

In other instances it may be, for one reason or another, that the existence or non-existence of an output signal is immaterial for certain combinations of input signals. From the standpoint of switching network design this situation is exactly the same as the previous one; in either case the response of the network to input signal combinations in question may be disregarded.

As an example, consider a switching network where the desired output signal in terms of the input signals may be represented by the expression, $\bar{A}B + A\bar{B}\bar{C}$, and assume that the combination of input signals represented by the elemental term, $A\bar{B}C$, will never be applied. For purposes of circuit simplification a good way to visualize the problem is to imagine that, if the combination were applied, an output signal would be caused. The representation of the switching function would then be $\bar{A}B + A\bar{B}\bar{C} + A\bar{B}C$, which can be reduced to $\bar{A}B + A\bar{B}$. The fact that this particular expression is not a correct representation of the desired switching function is of no consequence because it has been assumed that the combinations of input signals which would cause a discrepancy will never occur. As a variation in the example, assume that it is the combination, ABC, which is never applied to the input lines. In this case the expression which may be used is $\bar{A}B + A\bar{B}\bar{C} + ABC$. Although the A in the third term is superfluous, the expression is more complex than the original one; therefore the fact that ABC is never applied is of no help in this example.

As a further variation in the above example, assume that the input combination, AC, is never applied. This specification is the same as stating that neither $A\bar{B}C$ nor ABC will be applied. It may be imagined that, if AC were a valid input combination, an output signal would be generated. The resulting switching expression would then be

$$\bar{A}B + A\bar{B}\bar{C} + \bar{A}C = A\bar{B} + AB + AC.$$

In applications where three-input "and" switches are highly undesirable it is conceivable that this form of the switching network would be chosen in preference to the original form, which was $\bar{A}B + A\bar{B}\bar{C}$. However, in most applications the original form would probably be preferable.

From the nature of the above example it is apparent that, when making use of non-existent input combinations for finding network simplifications, they must be considered in all possible ways. More specifically, the non-existent input combinations must be expanded to their elemental form. Then attempts must be made to simplify the network by making use of each one combination in turn, each two input combinations, each three, and so on. In the above example the greatest simplification would be achieved when making use of the fact that $A\bar{B}C$ was non-existent even in the case when both $A\bar{B}C$ and ABC are non-existent. If, when represented in their elemental form there are N non-existent input combinations, a total of 2^N different switching expressions must be studied in order to find the most desirable one. While it is true that the task can be laborious when many input variables are involved, an experienced circuit designer can frequently eliminate many of the possibilities "by inspection."

A slightly more illustrative example is the switching function $\bar{A}B\bar{C}\bar{D} + A\bar{B}CD$ with the input combinations, $AC\bar{D}$ and $BC\bar{D}$, non-existent. If the switching network is altered so that it would produce an output in the presence of one or the other, or both, of these combinations, no simplification can be achieved (by most standards). However, when the non-existent terms are expanded to their elemental form, three different terms are found: $ABC\bar{D}$, $A\bar{B}C\bar{D}$, and $\bar{A}BC\bar{D}$. When the second and third of these terms, but not the first, are used, the switching function can be simplified to $\bar{A}B\bar{D} + A\bar{B}C$.

Miscellaneous Forms. In the general problem there is, of course, no requirement that the final solution be of the pure "and-to-or" or "or-to-and" variety. When multi-level switching networks are permissible it is frequently possible to find a more desirable arrangement through some miscellaneous form of network. As mentioned in a previous section, simple factoring of one or more of the variables will occasionally produce desirable results. In more complicated cases, significant improvements in the switching circuits can usually be achieved only through exercising considerable ingenuity; no general methods are known.

For an example, consider the switching function,

$$AB + A\bar{C}D + BC + BDE + \bar{A}DE + \bar{C}DE.$$

It is easy enough to show that this expression is equivalent to

$$(A + BC + DE)(B + \bar{A}E + \bar{C}D)$$

once this latter configuration has been found, but the finding of it is indeed a puzzle. Frequently, clues to arrangements of this type can be observed through a searching for similarities of variables in the various factors of the pure "or-to-and" form. Even then, a certain amount of cleverness and skill on the part of the circuit designer seems to be required.

Inverted Inputs Not Readily Available. It has been implied in all of the previous examples that the inverse of each variable has been available as an input signal when required. When the signals are generated by flip-flop type of circuits it is true that the inverse of any variable may be obtained merely by making a connection to the opposite side of the flip-flop. However, frequently more than just a connection may be required. In electronic circuits particularly, it may be necessary to intall a power amplifier between the flip-flop and the switching network. When the signals must be transmitted a substantial distance the fact that two wires are needed for each signal with its inverse may be an important disadvantage. Furthermore, in a calculator, signals may be obtained from many types of circuits other than flip-flops, and in these cases the inverted signals must usually be obtained through the use of inverters of some sort.

When inverters or other extra equipment must be used to obtain inverted signals, it becomes desirable to design switching networks with the minimization of inverters an objective as well as the minimization of "and" and "or" switches. Here the general problem becomes very complex because the most desirable network configuration in any given instance is dependent upon the relative importance of eliminating an inverter or other components. Also, the more subtle points such as number of levels of switching, the driving power of the input signals, switching speed, and possible non-existence of certain combinations of input signals must be taken into consideration.

Again, ingenuity on the part of the circuit designer is the primary requirement for finding the most desirable circuit. As an example, the switching function,

$$\bar{A}\bar{B}DE + \bar{A}\bar{B}C + \bar{A}\bar{C}DE + AB,$$

requires that three of the input variables be available in inverse form. Although it is difficult to find, the expression,

$$\overline{A + BC}\,(DE + C) + AB,$$

represents the same function, and only one inverter is required.

Multiple-Output Switching Networks. Multiple-output networks are, in general, even more remote from the cases for which systematic procedures are known for finding the most desirable arrangements. Of course, if each output signal may be generated by a separate network, all of the remarks made previously can apply. Also, when the networks are limited to pure "and-to-or" or "or-to-and" forms, obvious extensions of the previously described rules and procedures may be used as aids. The rules will help in finding terms or factors which may appear in the expressions for two or more of the output signals and which need not be duplicated in the physical circuitry.

The general problem of finding the most desirable switching circuit when two or more output signals are to be derived from one set of input signals can be a rather complex puzzle. A few practical examples are worked out in subsequent chapters. In particular, the full adders described in the chapter on binary addition and subtraction are examples of networks with three input signals and two output signals. In the chapter on decimal addition and subtraction a decimal adder operating in the 8, 4, 2, 1 code and involving nine inputs and five outputs is worked out in some detail. The method of analysis for this example was also used for deriving the 8, 4, 2, 1 doubler and quintupler described in the chapter on decimal multiplication and division. Since these and other examples meeting various specialized requirements can be found elsewhere in the text, none will be presented here.

Matrices. A certain category of multiple-output circuits deserve special mention because of their wide application. The circuits are known as "matrices" because they are sometimes drawn on paper (or occasionally even constructed physically) in an array of rows and columns which vaguely resemble mathematical matrices. A switching matrix is a switching network which has an output line corresponding to each possible combination of input variables; that is, an output signal appears on a separate wire for each elemental term composed of the input variables.

When only two variables are involved, the matrix as shown in Fig. 3-1 is almost trivial. An output signal is obtained on one of four separate output lines according to the four possible combinations of input variables. When A = 0 and B = 1, for example, a signal will be present on the line yielding $\bar{A}B$, but none of the others.

With three variables, either of the arrangements shown in Fig. 3-2 may be used. In (a) eight 3-input "and" switches are required, whereas in (b) twelve 2-input "and" switches are required. With diode switching of the type described, it happens that a total of 24 diodes is necessary in either case although when other types of switching components are employed one or the other of the arrangements may be preferable. The arrangement in (b) is sometimes called a "tree" or a "pyramid." Note that C appears as an input to a relatively large number of "and" switches when compared with B or A. This unequal loading of the input signals may be a disadvantage. The loading may be equalized somewhat by interchanging the B and C inputs in either the right-hand half or the left-hand half of the figure; it happens that this change does not affect the output functions.

For four or more input variables, obvious extensions of Fig. 3-2 may be made, but the number of diodes required for the two arrangements are no longer the same. With n input variables $n2^n$ diodes are required when the first type of arrangement is used, and $2^3 + 2^4 + \ldots + 2^{n+1}$ dioes are required for the "tree." The type of matrix which is most conservative in components, at least when diodes are used, is shown for four variables in Fig. 3-3. The variables are divided into two groups with one group including A and B and the other group including C and D. Four intermediate signal lines are derived from each group, and then these are combined in a set of sixteen two-input "and" switches which will yield a signal on one of sixteen output lines.

With five input variables an analogous array is used to minimize the number of components. In this case one group would contain two variables; the other would contain three with eight intermediate lines formed according to either of the arrangements shown in Fig. 3-2. The thirty-two output lines would then be obtained with a four-by-eight array of two-input "and" switches.

In general, with n input variables, the variables are divided into two groups with n/2 variables in each group when n is even and with (n + 1)/2 and (n - 1)/2 variables, respectively, when n is odd. Each group is divided into subgroups in a similar manner, and the subdividing is continued until all subgroups contain either 2 or 3 variables. The 2-variable and 3-variable subgroups are applied to switching networks of the types shown in Figs. 3-1 and 3-2, respectively. The subgroups are then combined with appropriate arrays of 2-input "and" switches.

In some applications the output signals from a matrix are used directly as implied in the preceding discussion. In other applications the matrix is used to "gate" an external signal (such as series of pulses) onto one of a multiplicity of signal lines. For three input variables probably the most obvious way of accomplishing the desired result is shown in Fig. 3-4 (a). Since the signal to be gated may be considered as another input to the switching network as a whole, a number of variations in the matrix are possible. Two variations are shown in Fig. 3-4 (b) and (c). Either of these arrangements requires less components than (a), but they have the disadvantage that the signal to be gated must pass through more "and" switches in succession, and with some types of components the delay might be excessive.

When a matrix involving four or more variables is necessary for the gating of the signal, extensions of the schemes shown in Fig. 3-4 may be readily worked out. One arrangement with four matrix variables, which combines the features of Figs. 3-3 and 3-4 (c), is given in Fig. 3-5. When choosing the most desirable arrangement for any given application, it should be noted that in some arrangements certain of the "and" switch inputs must pass the signal being gated while others need to respond only to the matrix switching signals. It may be that an "and" switch which must pass the gated signal is much more expensive than the other "and" switches. For this reason, the simple minimization of "and" switches is not necessarily the best criterion for judging the various possible network configurations.

Another important application of matrices is the selection of a signal from one on several different lines and applying this signal to a single output line. In this case the multi-output features of matrices substantially disappear and the notation used earlier in the chapter for single output circuits may be applied directly. For selecting one of eight signals, S_1 to S_8, by means of a matrix using three control signals, A, B, and C, the output may be expressed as

$$S = S_1 A B C + S_2 \bar{A} B C + S_3 A \bar{B} C + S_4 \bar{A} \bar{B} C$$

$$+ S_5 A B \bar{C} + S_6 \bar{A} B \bar{C} + S_7 A \bar{B} C + S_8 \bar{A} \bar{B} C.$$

This expression may be rearranged and factored in a number of different ways to produce new circuits. An example is

$$S = \left[(S_1 A + S_2 \bar{A})B + (S_3 A + S_4 \bar{A})\bar{B}\right] C$$
$$+ \left[(S_5 A + S_6 \bar{A})B + (S_1 A + S_8 \bar{A})\bar{B}\right] \bar{C}.$$

When four or more control signals are involved, the principle illustrated in Fig. 3-3 may be applied. If one of sixteen signals are to be selected, they may be entered as third inputs to the four-by-four array of "and" switches, and the sixteen outputs are then combined in an "or" switch.

In all of the matrix examples, when the number of outputs involved is not exactly 2^n, where n is some integer, it is necessary of course to choose a number, n, of control signals such that 2^n will be greater than the number of outputs. In these instances special arrangements can sometimes be found which will require less components than will be required through eliminating unused "and" and "or" switches in the more straightforward configurations.

Sequenced Signals. In some applications the sequence in which the various input signals are applied to a switching network is of consequence in the formation of the desired output signal. To make a switching network sensitive to the sequence of the applied signals it is necessary to employ feedback paths (storage).

As a simple example, consider two pulse-type signals, A and B, where it is desired that A not appear on the ouput and that B appear on the output line only in the event that A is applied prior to B. If it is known that A and B will never be applied very close together in time, it is sufficient to have A flip a flip-flop, the output of which is applied to an "and" switch that will control the passage of B. In the general case when the input signals may appear at substantially random times a more refined switching circuit is necessary. The difficulty arises from the fact that when A and B are applied close together in time, the flip-flop may be changing when B appears at the "and" switch. The amplitude of the output pulse might then be reduced by an unknown amount where it would be desired to have zero output or a full-sized pulse.

An arrangement which produces the desired result, except for a delay, is shown in Fig. 3-6. A flip-flop and an "and" switch are used as described in the previous paragraph, but the output of the "and" switch is used to flip a second flip-flop. The output of this second flip-flop is then combined in a second "and" switch with the delayed B signal. With this arrangement, the second flip-flop will either flip or not (no intermediate state is possible) regardless of the strength of the output from the first "and" switch. Consequently, the output pulse will be either full-sized or zero.

Several variations in the switching arrangement are possible which have various advantages and disadvantages depending upon the detailed requirements of the application. One type of variation worth noting is derived from the fact that, of the four possible combinations of stable states of the two flip-flops, one combination never exists. Specifically, the second flip-flop is never in the "1" state at the same time that the first flip-flop is in the "0" state. As a consequence, the two flip-flops may be replaced by a single configuration which has three stable states. The resulting arrangement is shown in Fig. 3-7. After a pulse is applied to the "reset" line, a signal appears at the output of the first inverter, but no signals appear on the output lines from the second or third inverters. If the B input pulse is applied prior to A, it will not pass either "and" switch. If A is applied prior to B, a signal will be caused to appear at the output of the second inverter. The B signal will then pass the first "and" switch and cause the array to exist in its third stable state. The second "and" switch will now be opened, and the delayed B pulse will appear on the output line. As before, the output pulse will be either zero or full-sized.

A requirement frequently encountered in the control portion of a calculator involves the starting and stopping of a uniform series of pulses by start and stop pulses which may be random in time. The basic problem involved, which is the causing of all output pulses to be either zero or full-sized, is substantially the same as when designing switching networks to respond to sequenced signals.

Three solutions to be problem are shown in Fig. 3-8. All are similar in their use of bistable storage elements, but the differences in the placement of the delay device create important differences in circuit operation as illustrated by the timing chart in the figure. In (a), all pulses occurring after start pulse and before the stop pulse will appear on the output line. However, they will be delayed because they must pass through the delay device. The amount of delay should be short relative to the time between successive pulses, but should be longer than the time required for the bistable device to change its state. In (b) the output pulses will not be delayed at all although the first pulse after the start pulse (the 2 pulse in the timing chart) may not appear on the output line. For the 2 pulse to appear, the start pulse must be applied at a time prior to the appearance of the 1 pulse at the A_1 and A_2 "and" switches. Similarly, the 6 pulse may or may not appear on the output in accordance with whether or not the 5 pulse arrives at the "and" switches prior to the application of the stop pulse. With the arrangement in (c) the first pulse to pass is the second one after the start pulse and the last one will be the first one after the stop pulse. In all three arrangements, any given pulse will either not pass at all or will pass with full amplitude regardless of the timing relative to the starting and stopping signals.

The time of application of signals of the steady-state type can be of consequence as well as when pulse signals are used. Flip-flops or other multistable configurations are employed in the same general manner. Because the problems that arise in practice are usually so "miscellaneous" in nature and because no organized methods of solution are known, the subject will not be carried farther.

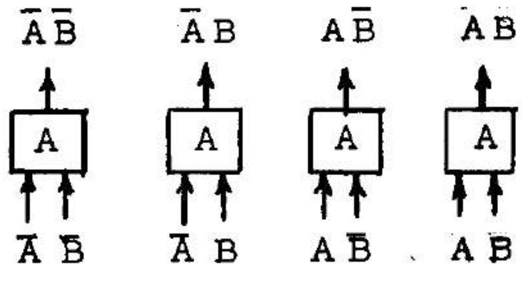
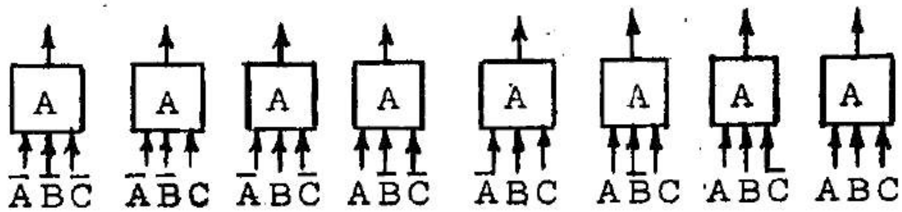
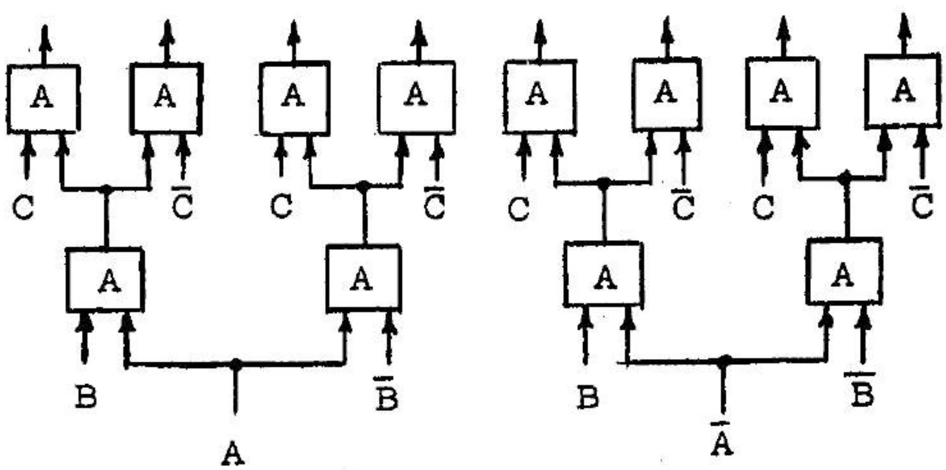


Fig. 3-1. Two-variable matrix.



(a)



(b)

Fig. 3-2. Three-variable matrices.

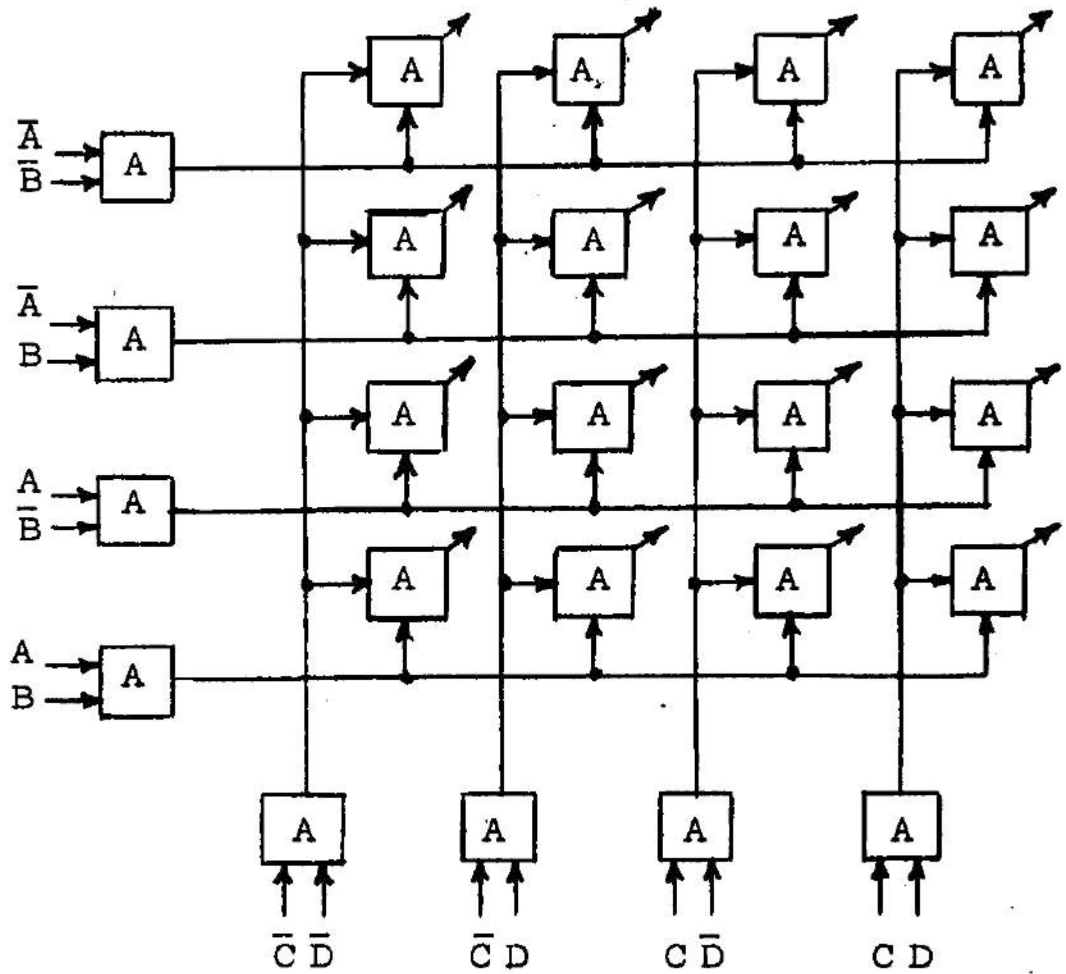


Fig. 3-3. One form of four-variable matrix.

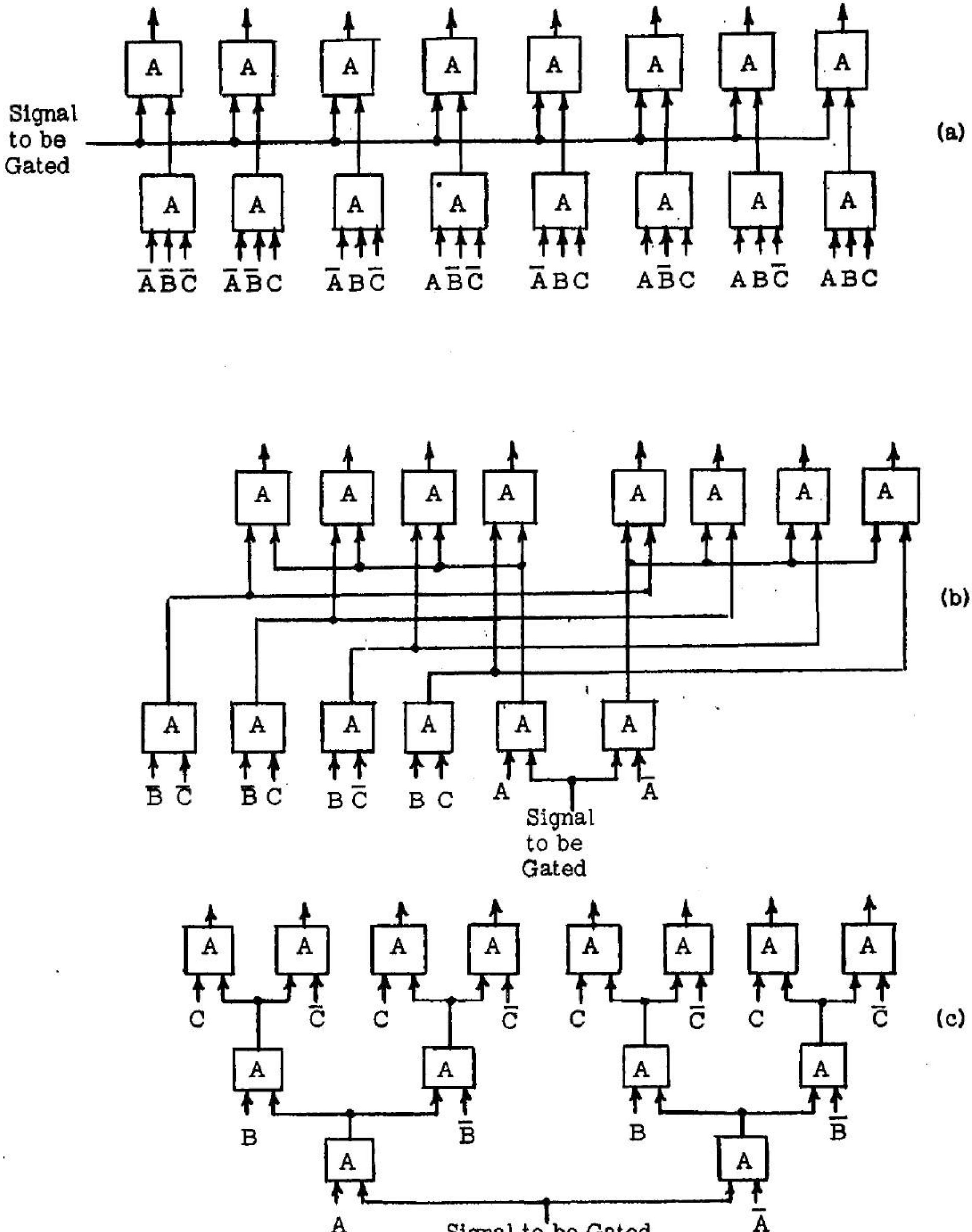


Fig. 3-4. Three-variable "gating matrices.

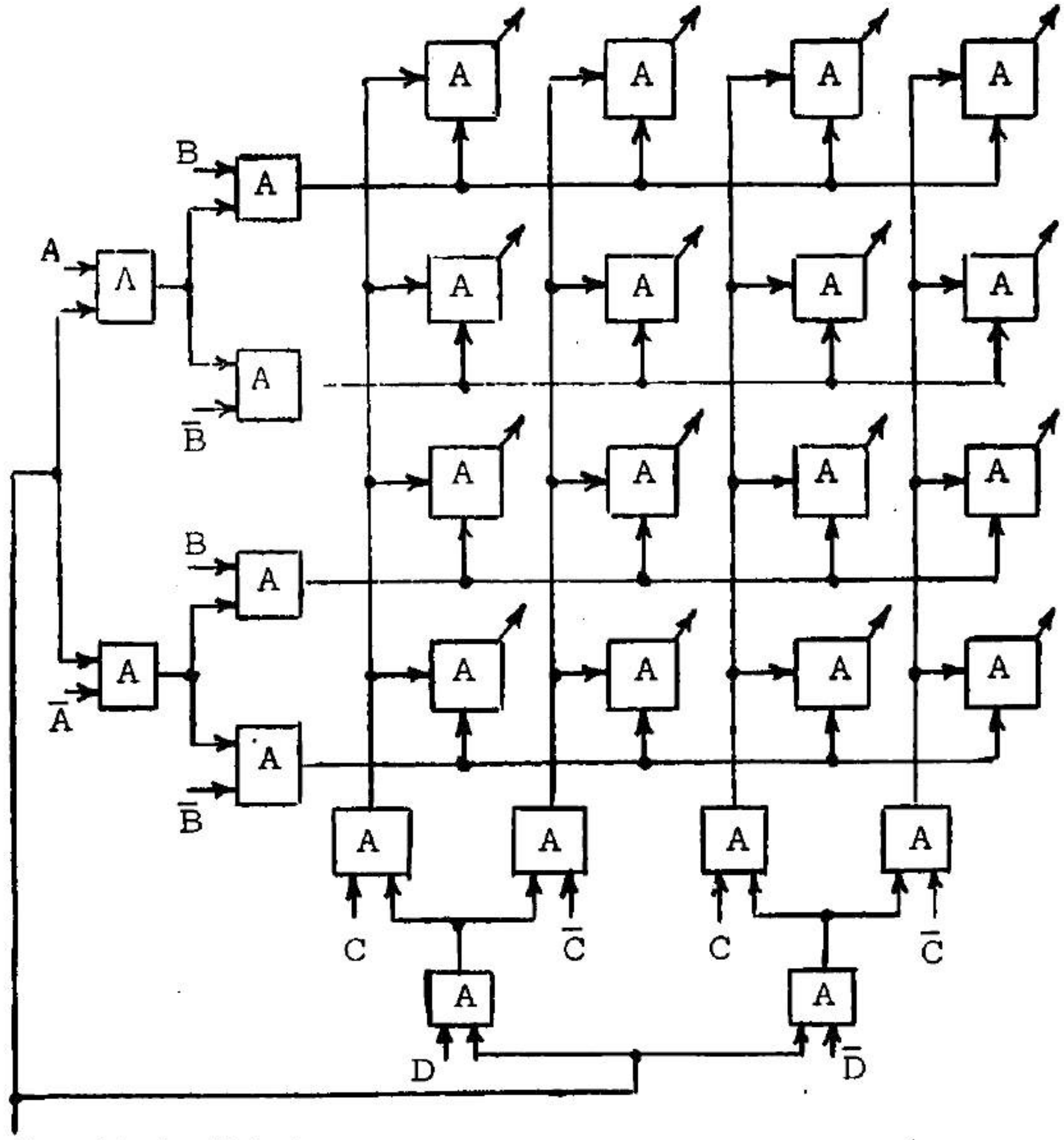


Fig. 3-5. One form of four-variable "gating" matrix.

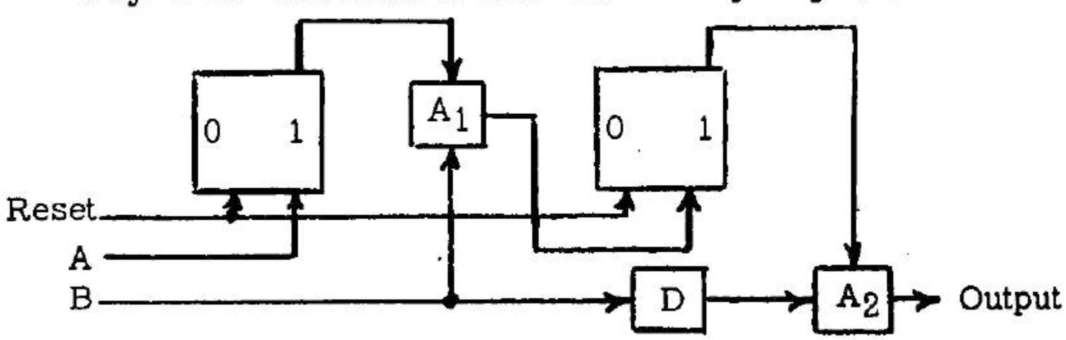


Fig. 3-6. Example of circuit responsive to sequence of signals.

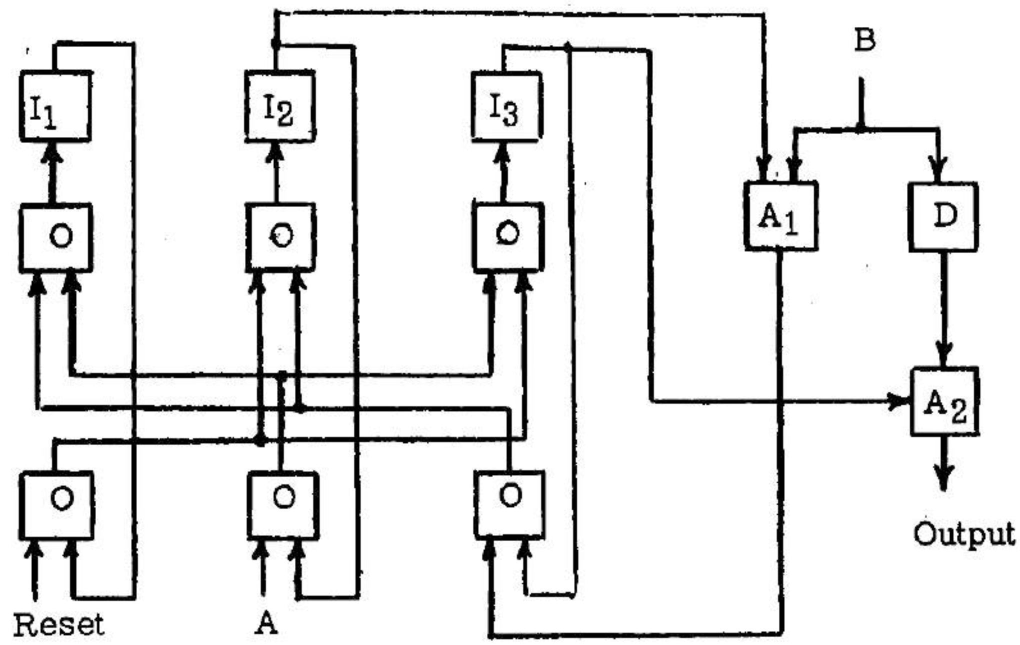


Fig. 3-7. A variation of Fig. 3-6.

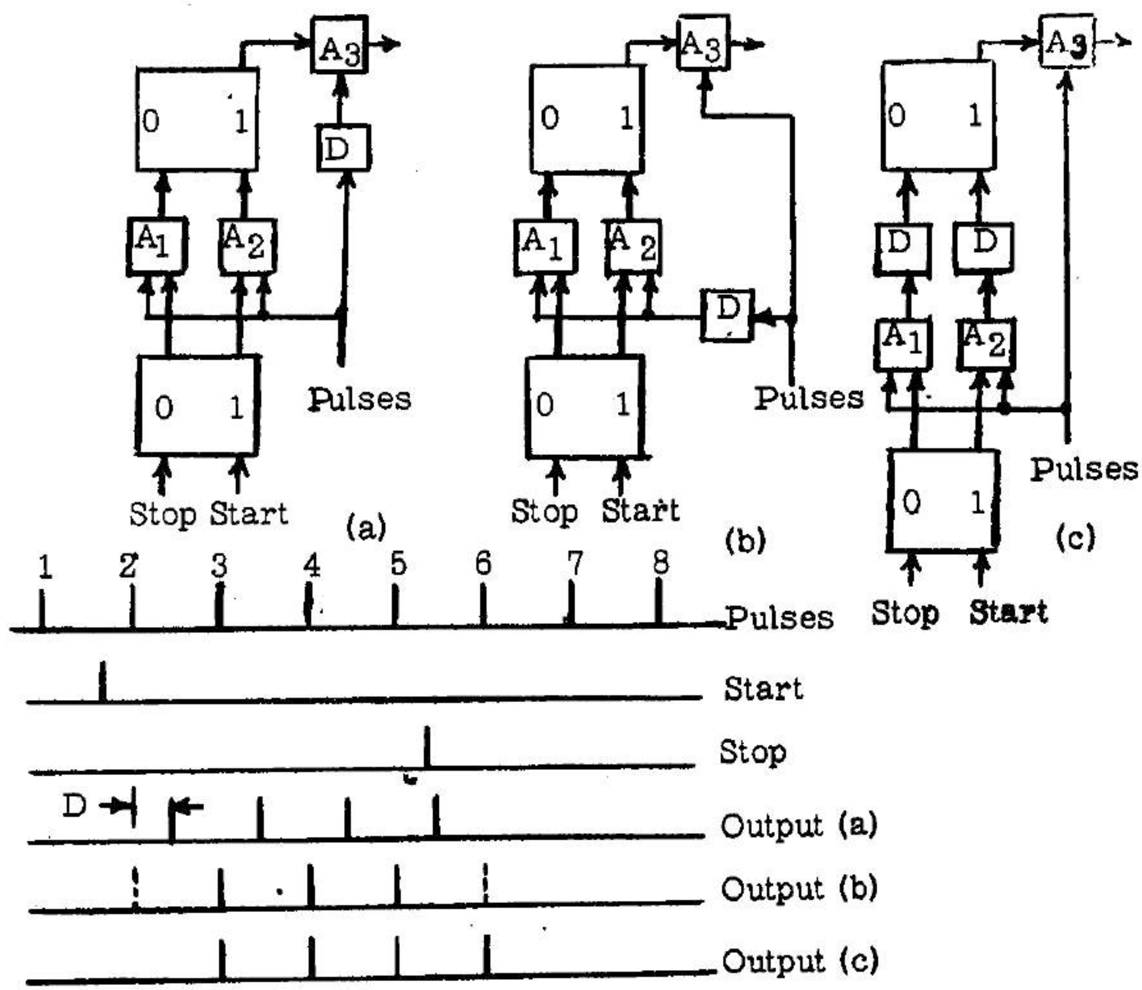


Fig. 3-8. Starting and stopping a series of pulses.