SPECIMEN TABLES
CALCULATED AND STEREOMOULDED
BY
THE SWEDISH
CALCULATING MACHINE.
SPECIMENS

OF

TABLES,

CALCULATED,

STEREOMOULDED, AND PRINTED

BY

MACHINERY.

LONDON:
LONGMAN, BROWN, GREEN, LONGMANS, AND ROBERTS,
PATERNOSTER ROW.
1857.
TO

CHARLES BABBAGE, ESQ., F.R.S.,
&c. &c. &c.,

THese PAGES ARE DEDICATED,

BY HIS SINCERE ADMIRERS,

GEORGE AND EDWARD SCHEUTZ.
PREFACE.

To those to whom this work is addressed, it is unnecessary to point out the manifold advantages which have been derived in modern times from the various Numerical Tables now in the hands of the cultivators of science. The gigantic efforts that have led to the many important discoveries which have adorned the present century, would never have been made if these Tables had not existed. Astronomers, Navigators, Engineers, Actuaries, and indeed labourers in every department of science and the useful arts, could have made but little progress in their several vocations, had it not been for the resources supplied to them by those men of genius and of patience, who devoted their energies to this beneficial object: and when we contemplate the number and range of these Tables, and reflect upon the difficulties surmounted in their production, the time consumed, the labour employed, the capital expended, we cannot but regard with heartfelt gratitude and admiration those men who, without hesitation, and with an indomitable perseverance, devoted so many years of their lives to this irksome and herculean task; and that too, be it remembered, at a time when very few of their contemporaries could estimate the value of the work they were accomplishing. Such men may indeed be called the Martyrs of Science!

While acknowledging, however, the deep debt of gratitude which is due to the intellect, the industry, and the self-sacrifice of a Napier or a Briggs, we must still admit that, notwithstanding the pains taken to insure accuracy, much still remains to be done. There is scarcely any numerical table, or perhaps it may more truly be said that no one table has ever been published, which does not contain errors more or less numerous; and from a careful examination of a great number of existing tables, Mr. Babbage long ago arrived at the conviction that by
employing the means hitherto resorted to for constructing them, it was impossible ever to obtain more than an approximate degree of correctness. He accordingly conceived the project of constructing an engine, which, by mechanism alone, would not only compute tabular numbers, but at the same time also impress its results upon plates of copper or other material, from which they might be immediately printed on paper, so as entirely to supersede the use of ordinary types.

This happy idea of combining the act of mechanical computation of series of numbers with that of simultaneously printing them, was perfectly original; although, when once propounded, it appears obviously to be the only mode by which, at a moderate cost, and with an infinite saving of labour, numerical tables can be obtained, with the positive certainty of their being wholly exempt from error.

Such was the origin of the Calculating Machine, invented by Mr. Babbage, and which, from its operating by the successive additions of several orders of differences derived from any numerical series, having all its terms connected by some analytical law, he denominated the Difference Engine.

The only analogous contrivances which had been effected, or even proposed, prior to Mr. Babbage's Difference Engine, were designed for the performance merely of single arithmetical operations of the most elementary kind, such as addition, subtraction, multiplication, and division—the very operations which numerical tables are intended to supersede, by sparing the inquirer the trouble of performing them either by the labour of the head or the manipulations of a machine, and presenting the results to his eye in the columns of a printed page.

The Difference Engine of Mr. Babbage had from time to time been cursorily noticed in several periodicals, when a circumstantial and elaborate disquisition on its merits and construction appeared in the Edinburgh Review for July, 1834. It was from the perusal of this article that Mr. George Scheutz, at that time the editor of a technological journal in Stockholm, derived the first conception of constructing a machine for effecting the same purpose as that of Mr. Babbage, namely, that of calculating and simultaneously printing numerical tables.* But after he had satisfied himself of the practicability of the scheme, by constructing various models, composed of wood, pasteboard, and wire, he postponed to a future period the further prosecution of the design.

Three years afterwards, in the summer of 1837, his son, Mr. Edward Scheutz, at that time a student in the Royal Technological Institute at Stockholm, offered to his father to construct a working model in metal, on condition that he might have the use of a particular work-room in his father's house, as well as of a lathe and other necessary tools. With these implements he commenced the work, and during the summer vacation had so far completed it, as to demonstrate the feasibility of the scheme, and its applicability to the practical purposes for which it was designed.

Matters being so far advanced, his father made application to the Government on the 30th of October in the same year, for their sanction and assistance in the completion of the plan. To this application a negative answer was returned on the 21st of February in the following year.

Not discouraged by this refusal, Mr. E. Scheutz continued to employ his time on the model, holding frequent discussions with his father relative to the improvements in the original conception; sometimes the one and sometimes the other suggesting expedients for overcoming the difficulties that presented themselves. After many trials and many alterations, the calculating apparatus was, in the year 1840, so far completed, that it correctly calculated series with terms of five figures and one difference, also of five figures. On the 29th of April, 1842,

* The article in the Edinburgh Review above referred to also suggested to Mr. Alfred Deacon the contrivance of a difference engine for the same purpose, but on a plan of construction very different. This machine is still in existence, although not in a finished state; it was intended to calculate consecutive numbers to twenty figures, with three orders of differences; but, on account of the expense, printing the result was not contemplated.
the model was extended so as to calculate similar series with two and three orders of differences.

The following year, 1843, the printing apparatus, and all the other parts of the model, were in readiness for the inspection of the Royal Swedish Academy of Sciences. After several trials, a certificate was obtained on the 18th September, 1843, signed by Baron Jacob Berzelius, Secretary of the Academy; N. H. Selander, Astronomer of the Academy; and E. B. Liljehöök, R. N., Professor of Natural Philosophy in the Royal Military Academy at Marieberg.

From this certificate the following passages are extracted:

"The apparatus in question is composed of three parts.

"1st. The Calculating Machine.—It cannot compute series of a higher degree than the third, nor does it give complete terms exceeding five figures; but in the nature of the mechanism, there is nothing to prevent its extension to the working of series of any degree whatever, and to terms of as many figures as the purpose may require.

"2nd. The Printing Machine.—Every term given by the calculating apparatus is expressed by printed figures, closely arranged in lines, as in a printed table, the lines being impressed on some softer material, adapted to receive galvanoplastic or stereotyped copies. All the lines succeed each other very correctly in the same vertical column.

"3rd. The Numbering Machine.—With the printing machine, another apparatus is combined, which prints the arguments before every term. The machine is put in motion by turning the handle of a winch, by means of which, and without further manipulations, the calculation, as well as the printing and arranging of figures and lines, are effected."

The inventors sought for orders in various countries, making use of the above certificate as a recommendation; but, meeting with no success, the model remained shut up in its case during the ensuing seven years.

But in the year 1850, December 26th, another inspection was made by a committee of the Royal Academy of Sweden, consisting of General Baron Fabian Wrede, Chief of the Royal Military Academy at Marieberg; and the late P. A. Wallmark, R.N., Director of the Royal Technological Institute of Stockholm.

In the beginning of the following year, 1851, January 28, Mr. George Scheutz made a fresh application to the Government, to obtain from them the means of carrying the plan into full execution, by the construction of a larger and still more improved machine. His request was first referred to the judgment of the Academy of Sciences, which then advised the Government to consent to the application, because, as their Report says, "the Messrs. Scheutz had devoted many years of labour to this invention, which, without the least doubt, would be useful to science; and had incurred expenses, which, with relation to their resources, were very considerable; and that Mr. Scheutz, sen., had for a long time, without assistance from the public, published many useful works in his native country, relating to industrial progress."

The decision of the Government, given on the 29th April, was, that there were no public funds at their disposal for the object in question.

In the Diet of 1851, Mr. A. M. Brinck, a merchant, and member of the Diet for Stockholm, moved that a national recompense should be given to the inventors. The motion was acceded to, on condition that the money should be paid to Messrs. G. and E. Scheutz after his Majesty the King had, on due examination, given it as his opinion that the machine was completed and answered its purpose. The inventors now sought to anticipate the reward, in order to have the means of completing the invention, and their petition was granted on the 24th October, 1851, on condition that the money was to be refunded if the machine was not completed before the end of the year 1853, or if, when completed, it was not found to answer its purpose.

In consequence of these limitations, the inventors were obliged, in the first instance, to procure a guarantee for the repayment, in case of failure. Partly by their own exertions, and partly by the kind assistance of some members of the
Swedish Academy, they were fortunate enough to collect a list of subscribers, each of whom rendered himself responsible for that part of the amount which was annexed to his name. The names of the subscribers are found in the subjoined note; and the author of this little book embraces this opportunity of expressing to them, in the names of the inventors of the machine, their gratitude for the means thus afforded of bringing it into its present improved state. Without such patronage, indeed, the machine would probably never have been completed, nor these tables produced.

The first working drawing of the new machine was finished on the 1st of February, 1852, and the machine itself was completed before the end of October, 1853.

The machine thus completed differs in many parts from the first model, which was confined in its range to the decimal scale, and could neither produce tables containing hours or degrees, minutes and seconds, nor effect the requisite corrections in the last printed figure. For the suggestion of these important improvements, the inventors are indebted to General Baron Fabian Wrede, who was one of the most zealous promoters of the undertaking. The machine was constructed in conformity with the drawings of Mr. Edward Scheutz, and under his immediate superintendence, at the manufactory of Mr. C. W. Bergström, at Stockholm; and when all the parts were finished, and finally put together, the new machine performed its work so perfectly, that the first trials required no alteration whatever. In the Report to H. M. the King of Sweden, the Royal Academy not only declared that the machine answered its purpose, but at the same time expressed its conviction, founded on the evidence of facts, that the expenditure for the completed machine had far exceeded the sum awarded as the national recompense. His Majesty then proposed to the Diet to give

| E. F. Berg | J. Falkenbolm | P. W. Lundgren |
| C. F. Bergström | N. Graffström | B. O. Nylander |
| J. W. Bergström | L. J. Hierta | P. Siljestrom |
| C. Craswell | G. Lallerstedt | P. E. Swedboom |
| Eklund | Lovén | J. Wahlström |

the inventors an additional recompense, of the same amount as the former one, which proposition was agreed to, the inventors receiving an official notice of this decision in their favour, from the Royal Academy of Sciences, on the 11th of August, 1854.

It was during the three last months of the same year, that the inventors visited England and France. On their arrival in London, they met with one of their countrymen, Count P. A. Sparre, by whom they were introduced to Messrs. Bryan Donkin & Co., Civil Engineers, who allowed the two foreigners to exhibit the machine at their manufactory in Bermondsey. It was here that they became acquainted with Mr. William Gravatt, F.R.S., Civil Engineer, who took a very warm interest in the invention, gave notice to the Royal Society of the arrival of the machine, and showed and explained it to several of the most eminent members of that learned body—namely, Colonel Edward Sabine, Doctor P. M. Roget, Professors Faraday, Hind, Stokes, Wheatstone, Miller, and several others.

After the departure of the inventors, the Royal Society, at Mr. Gravatt’s request, most kindly allowed the machine to remain in one of their apartments in Somerset House, where Mr. Gravatt had also the kindness to work it, and to explain its mechanism on many occasions to the Fellows of that illustrious Society and other visitors. On the 29th of January, 1855, it was honoured by the inspection of His Royal Highness Prince Albert.

The machine was in the same year conveyed under the care of Count Sparre, to Paris, and placed in the Great Exhibition, where Mr. Gravatt again had the kindness to work and explain it to many scientific gentlemen. The jury to which it was referred, composed of distinguished persons of all nations, after a full examination, unanimously awarded the Gold Medal to the inventors, who received it publicly through the hands of His Royal Highness the Prince Charles, in the Royal Palace of Stockholm, on the 21st of April, 1856. Some time previously, Mr. Scheutz, sen., was elected a Member of the Swedish Academy of the Third Class for the Science of Mechanics; and
on the 28th of April he was made a Knight of the Order of Wasa.

A letter from Mr. Bryan Donkin, of the 5th June, 1856, induced Mr. E. Scheutz again to make a short visit to London, where he had the gratification of meeting those friends whose interest in the invention had remained unabated.

By the advice of Mr. Gravatt, to whom the inventors were already under such deep obligation, it was resolved that the machine should again be brought over to England, and, as a last attempt, that an endeavour should be made to obtain an acknowledgment of the practical usefulness of the invention by the publication of a collection of specimens of numerical Tables computed and printed by the machine. Mr. Gravatt had also the kindness to set apart a room in his own house for the performance of this work, and offered to promote the undertaking in every way. The inventor, now animated with fresh hopes, returned to France in order to fetch the machine, and it was during his stay there that he had the honour of showing the engine at work in the Imperial Observatory.

After Mr. E. Scheutz's return to London, as soon as the machine was put in order, and he had consulted Mr. Babbage and Mr. Gravatt with regard to the specimen Tables to be printed, this work was commenced, the calculations for setting the machine being made partly by Mr. Gravatt, according to formulae of his own, and partly by Mr. E. Scheutz, after formulae which General Baron Fabian Wrede had the kindness to give him on his departure from Sweden.

Meanwhile, what had been passing in London and in Paris had not escaped the notice of Professor B. A. Gould, of the Dudley Observatory, Albany, in the United States, who, conceiving the possibility of procuring purchasers for the machine in America, exerted himself in the most liberal and enlightened spirit to accomplish this object. The inventors had so many times seen their hopes blighted, that they did not expect that anything decisive would result from this new attempt; but they found themselves mistaken, for, after some time, they received information from Professor Gould that the terms they had pro-

posed had been accepted; the machine, therefore, now belongs to the Dudley Observatory, at Albany, being a gift to that Observatory from an enlightened and public-spirited merchant of that city, John Fr. Rathbone, Esq.

In the course of their career, the inventors have had to go through many difficulties; but they have also been amply rewarded by the friendship and sympathy, which so many good and noble-minded men have manifested towards them, and by the certainty they now possess, that the object for which they have so long and so severely toiled, will not be lost. They believe, also, that if this limited collection of specimens should be deemed not sufficient to prove the practical usefulness of the invention, it will soon be followed by other proofs of greater importance, printed by the astronomers of the Western World; and that the individual to whom they have with gratitude and regard dedicated this little work, will then be further known for what he truly is—namely, one of the benefactors of mankind, and one among the noblest and most ingenious of the sons of England.

The drawing here given, is from an engraving kindly lent by the proprietor of the Illustrated London News, and exhibits the machine at a time when it has just completed the operation of stereotyping a result extending to eight places of figures.

It may be as well to premise that the size of the whole machine, when placed on its proper stand and protected by its cover, is about that of a small square pianoforte.

The calculating portion of the machine, which appears in the front of the drawing, consists of a series of fifteen upright steel axes, passing down the middle of five horizontal rows of silver-coated numbering rings, fifteen in each row, each ring being supported by, and turning concentrically on its own small brass shelf, having within it a hole rather less than the largest diameter of the ring. Round the cylindrical surface of each ring are engraved the ordinary numerals from 0 to 9, one of which, in each position of the ring, appears in front, so that the successive numbers shown in any horizontal row of rings may be read from left to right, as in ordinary writing.

The upper row exhibits the number or answer resulting from
the calculation to fifteen places of figures, the first eight of which the machine stereotypes. The numbers seen on the second row of rings constitute the first order of differences, also to fifteen places of figures, if that number be required; and the third, fourth, and fifth row of rings, in like manner, exhibit the second, third, and fourth orders of differences. Any row can be set by hand, so as to present to the eye any number expressed according to the decimal scale of rotation; such as the number 987654321056789, the first eight figures of which, if in the uppermost row, would, on being calculated by the machine, be immediately stereotyped. But by simply changing a ring in each of two of the vertical columns, the machine can be made to exhibit and to calculate numbers expressed in the mixed senary system of notation, as in that of degrees, minutes, seconds, and decimals of a second. Thus, for instance, if the result 874324677356402 were indicated in the upper row of rings, it would be stereotyped 87 degrees 48 minutes 24.69 seconds. While this process is going on, the argument proper to each result is at the same time also stereotyped in its proper place; nothing more being required for that purpose than to set each row of figure rings to differences previously calculated from the proper formula, and to place a strip of sheet lead on the slide of the printing apparatus; then, by turning the handle (to do which requires no greater power than what is exerted in turning that of a small barrel-organ), the whole table required is calculated and stereomoulded in the lead. By this expression is meant that the strip of lead is made into a beautiful stereotype mould, from which any number of sharp stereotype plates can be produced ready for the working of an ordinary printing press. At the average rate of working the machine, 120 lines per hour of arguments and results are calculated and actually stereotyped, ready for the press. It is found on trial that the machine calculates and stereotypes, without chance of error, two-and-a-half pages of figures in the same time that a skilful compositor would take merely to set up the types for one single page.

The appearance of the tables would have been much improved had the machine been provided with better steel figure-wheels, the reason for not adopting which alteration may be gathered from the little historical sketch herein given.

In order to limit the size of this little book, the specimens given are neither so numerous nor so extensive as was at first intended. A great number of tables were suggested, but they all so evidently came within the power of this machine, and as mere repetition was to be avoided, the short specimens Nos. 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11 were considered sufficient as examples of tables perfectly easy with this machine, which happens to be limited to four orders of differences.

Specimen 12, the radius vector of Venus, and 13, the Sun's longitude for every twenty-four hours, are examples of calculations also easily within the power of four differences, but of which the number of terms obtainable at such wide intervals at one setting is considerably less than in the first-mentioned specimens. If, however, we wanted the Sun's longitude, or the radius vector of Venus, or any similar table, for intervals of every three hours for instance, it is evident that the machine could produce at each setting a table of eight times the extent of the specimens given.

In specimen 14, the logarithm of the Earth's radius vector, and still more in specimen 15, the logarithm of the distance between the Earth and Mars, in consequence of the wideness of the intervals, the want of more than four orders of difference, or the addition of some such contrivance as Mr. Babbage has alluded to in his letter to Sir H. Davy, July 3rd, 1823, begins to be felt. Should, however, five places of decimals be considered sufficient, double the number of terms could be produced in one setting in each of the specimens 14 and 15; or, if the intervals were smaller, the same remark would apply to these.

specimens as to specimens 12 and 13; and it must be borne in mind, that, however short may be the range in this machine at any one setting, the answers are calculated indifferently in either the decimal or in the mixed senary scales, and stereomoulded at the same time without chance of error.

The following is an abstract kindly furnished by Mr. Gravatt of his own manner of considering and of working this machine.

If \( u \) be any function of \( x \), and we set in the machine its initial value \( u \) and the finite differences \( \Delta^1 u \), \( \Delta^2 u \), \( \Delta^3 u \), \( \Delta^4 u \), \( \Delta^5 u \), \( \Delta^6 u \), in the order shown in the left-hand column below:

\[
\begin{array}{ccccccc}
0 & u & & & & & \\
1 & & & & & & \\
2 & & & & & & \\
3 & & & & & & \\
4 & & & & & & \\
5 & & & & & & \\
6 & & & & & & \\
\end{array}
\]

Then in the first half-stroke the machine will simultaneously add the even differences to the odd differences immediately above them, giving the new odd differences \( \Delta^3 u \) and \( \Delta^5 u \). At the next half-stroke it will add these new odd differences to the old even differences, thereby forming \( \Delta^3 u \) and \( u \) and the machine will appear arranged as shown in the second column. In like manner will be formed the third, fourth, \&c. columns, and if \( \Delta^4 u = \Delta^5 u = \Delta^6 u = \&c. \), that is if the fourth difference is constant, the machine will go on calculating the successive values of \( u \) as long as we please to turn the handle.

We see, therefore, that this machine is at least capable of tabulating any function of \( x \) in which the fourth difference is constant.

As an example, let us take \( u = 1 \), \( \Delta^1 u = 1 \), \( \Delta^2 u = 2 \), \( \Delta^3 u = 0 \), \( \Delta^4 u = 0 \); the machine when set would appear as in the left-hand column below, where the third and fourth differences being zero, are, for the sake of simplicity, omitted.

\[
\begin{array}{ccccccc}
1 & 4 & 9 & 16 & 25 & 36 & \&c. \\
1 & 3 & 5 & 7 & 9 & 11 & \&c. \\
2 & 2 & 2 & 2 & 2 & 2 & \&c. \\
\end{array}
\]

Here at the first half-stroke 2 is added to 1 giving 3, and at the second half-stroke this 3 is added to 1 giving 4. Again, 2 is added to 3 giving 5, and this 5 is added to 4 giving 9, and so on; the machine as thus set producing the squares of the natural numbers for ever.
Again, if we set the machine to the numbers shown in the left-hand column below,

\[
\begin{array}{cccc}
1 & 16 & 81 \\
1 & 15 & 65 & 175 \\
14 & 50 & 110 \\
12 & 36 & 60 & 84 \\
24 & 24 & 24 & 24
\end{array}
\]

24 will be added to 12 and simultaneously 14 to 1, giving respectively 36 and 15, and then 36 will be added to 14 and simultaneously 15 to 1, giving respectively 50 and 16, and so on; the machine as thus set producing the squares of the squares, or the fourth powers of the natural numbers as long as we please to turn the handle.

In the last example the machine is supposed to be stopped at a half-stroke, with only the odd differences 84 and 175 in the fourth column. Now, if we re-set the machine (changing the sign of these odd differences) as shown in the left-hand column below,

\[
\begin{array}{cccc}
81 & 16 & 1 \\
110 & 65 & 14 & -15 \\
-84 & -60 & -36 & -12 \\
24 & 24 & 24
\end{array}
\]

we shall have \(24 + -84 = -60\), and simultaneously \(110 + -175 = -65\) for the first half-stroke, and \(-60 + 110 = 50\), and simultaneously \(-65 + 81 = 16\) for the second half-stroke, and so on. That is, if we in this way change the signs of the odd differences, the machine will, so to speak, go backwards, and that as long as we please to turn the handle.*

A negative number is expressed in the machine by its complement; thus, if to 56789, we wish to add, say minus 67 (that is, in fact, to subtract 67), we set the machine to add 99933 to 56789, thus:

\[
\begin{array}{ccc}
56789 \\
less & 67 & \text{added to} & 99933 \\
gives & 56722 & \text{gives} & 156722
\end{array}
\]

but as in the machine there is purposely no figure-wheel to receive the left-hand 1, it is thrown (so to speak) into the air, and the machine shows and stereomoulds 56722 as it ought to do.

* In this way we can always make the machine itself show us the differences with which it was set.

This particular machine is capable of expressing \(u\) and its differences as far as the fourth order, to fifteen places of figures; but we may have to deal with differences of more than fifteen places, although we only require four, five, six, seven, or at most eight places of figures, to be printed as the tabular value of \(u\).

Now if the machine be worked, so as to give very many values of \(u\), it is evident that the omission of the 16th, &c., places of figures might begin to tell in the tabulated result.

To avoid this, it is necessary to know how many values of \(u\) we may obtain without an error, in the lowest figure of the printed result.

For this purpose, if we, considering what has gone before, imagine the machine set, as shown in the first column below, putting in only one value \(\Delta\), and that in the fourth order of differences,

<table>
<thead>
<tr>
<th>1st</th>
<th>2d</th>
<th>3d</th>
<th>4th</th>
<th>5th</th>
<th>(n)th Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1(\Delta)</td>
<td>5(\Delta)</td>
<td>15(\Delta)</td>
<td>(\cdots) (\frac{n-3}{2}) (\frac{n-1}{2}) (\frac{n-1}{2}) (\Delta)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1(\Delta)</td>
<td>4(\Delta)</td>
<td>10(\Delta)</td>
<td>(\cdots) (\frac{n-3}{2}) (\frac{n-1}{2}) (\frac{n-3}{2}) (\Delta)</td>
</tr>
<tr>
<td>0</td>
<td>1(\Delta)</td>
<td>3(\Delta)</td>
<td>6(\Delta)</td>
<td>10(\Delta)</td>
<td>(\cdots) (\frac{n-3}{2}) (\frac{n-1}{2}) (\Delta)</td>
</tr>
<tr>
<td>0</td>
<td>1(\Delta)</td>
<td>2(\Delta)</td>
<td>3(\Delta)</td>
<td>4(\Delta)</td>
<td>(\cdots) (\frac{n-3}{2}) (\Delta)</td>
</tr>
</tbody>
</table>

we should, in the manner in which the machine combines the differences (that is, always in simultaneously formed couples), obtain the 2nd, 3rd, 4th, 5th, and \(n\)th columns, the coefficients of \(\Delta\) being necessarily and obviously *figurate numbers* of the various orders (in this case up to the 5th order) but with the 2nd and 3rd *figurate numbers* set forward one term, and the 4th and 5th *figurate numbers* set forward two terms—the law for any machine taking in any order of differences being evident.

We immediately see that if we were to set in such a machine \(\Delta^2\Delta\frac{n-1}{2}\Delta^2\Delta\frac{n-3}{2}\Delta\), putting \(P_{n}\) for the \(n\)th printed number we shall get

\[P_{n} = \Delta^2 + \frac{n-1}{2} \Delta^2 + \frac{n-3}{2} \Delta^2 + \frac{n-1}{2} \Delta \]

Whence we see that if the error from leaving out the 16th, &c., figures, be \(a\) in the first difference, and \(\beta\), \(\gamma\), \(d\), respectively, in the 2nd, 3rd, and 4th differences (putting \(e\) the error in \(P_{n}\)), we should have \(e\) always less than \(a\) \(\frac{n}{2}\) \(\beta\) \(\frac{n}{6}\) \(\gamma\) \(\frac{n}{24}\) \(d\).
Now as we may always set the machine to \( u + \frac{1}{4} e \) instead of \( u \) (thereby halving the effect of any error) we may, even when working with all the four differences, put practically

\[
e = \frac{n^2}{48} \* \]

If \( m \) be the number of places required in the table, it is usual not to allow a greater error than \( \pm 5 \) in the \( m + \frac{1}{15} \) place. Now, the greatest value \( s \) can have is (less than) \( 5 \times 10 \), whence we readily see that we may put \( n^2 = 48 \times 10 \). So that if, for instance, we required the machine to print 8 places correctly, we have \( n^2 = 48 \times 10^9 = 480000000 \) or \( n = 148 \) for the number of values of \( u \) that we can print in each direction.

If we wish to print only 7 places correctly, we have \( n^2 = 48 \times 10^6 \) or \( n = 283 \).

If we wish to print only 5 places correctly, we have \( n^2 = 48 \times 10^4 \) or \( n = 328 \), so that in this case we may print about 800 terms for each forward and for each backward working of the machine, or together about 1600 terms.

The following method of finding proper differences with which to set the machine (found on the 5th lemma of the 3rd book of the Principia) is general, and extremely easy in practice. Let \( u_1 = u_0 + ax + bx^2 + cx^3 + dx^4 \) now putting \( x \) successively \( 0 \pm 1 \) and \( 0 \pm 2 \) we get

\[ u = u + 2a + 4b - 8c + 16d \]
\[ u = u + a + b - c + d \]
\[ u = u + a + b + c + d \]
\[ u = u + 2a + 4b + 8c + 16d \]

1st Differences. 2nd Differences.

\[ a = 2b - 6c + 14d \]
\[ a = b + c - d \]
\[ a = b + c + d \]
\[ a = 3b + 7c + 15d \]

3rd Differences. 4th Differences.

\[ 6c - 12d \]
\[ 6c + 12d \]

\[ f = \frac{1}{4} \Delta^4 u \]
\[ e = \frac{1}{4} \Delta^4 u + \frac{1}{4} \Delta^4 u \]
\[ d = \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]

\[ c = \frac{1}{4} \Delta^4 u + \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]
\[ b = \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u + \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]

\[ a = \Delta^4 u + \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]

Now, if \( n = 1 \) be the number of terms to be interpolated between \( u_1 \) and \( u_{n-1} \) and \( u_{n+1} \); putting \( \frac{1}{n} = x \) we get immediately

\[ \Delta^4 u \]
\[ \Delta^4 u \]
\[ \Delta^4 u \]
\[ \Delta^4 u \]

With which differences we can set the machine and tabulate forwards from \( u \) through \( u_1 \) to \( u_{n-1} \), and by changing the sign of the odd differences in the manner before shown, we can tabulate backwards from \( u \) through \( u_1 \) to \( u_{n-1} \), and this without knowing even the form of the function we are tabulating, it being sufficient that we have five values taken at equal intervals.

Now \( u \) the function of \( x \), we may have to consider, may not have the \( 4^2 \), nor indeed any order of differences constant, and to see the consequences of this let us compare (as in most cases will be amply sufficient) the value of the \( u_{n-1} \), derived from four order of differences (which I write \( u_{n-1} \)) with the value of \( u_{n-1} \), derived from six order of differences (which I write \( u_{n-1} \)).

In the equation \( u_n = u_0 + a + b + c + d + e + f + x^4 \), putting \( x \) successively \( 0 \pm 1, \pm 2, \pm 3 \), in the manner before shown, we get

\[ f = \frac{1}{4} \Delta^4 u \]
\[ e = \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]
\[ d = \frac{1}{4} \Delta^4 u + \frac{1}{4} \Delta^4 u \]

\[ c = \frac{1}{4} \Delta^4 u + \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]

\[ b = \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u + \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]

\[ a = \Delta^4 u + \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u - \frac{1}{4} \Delta^4 u \]

* The way I use the above formulas in practice, is to begin, at the right-hand side of properly ruled paper, with the 4th differences, performing the calculations line by line in the order shown in the example given in page 8. In order to enable us to work always forwards from \( u \) through \( u_1 \) to \( u_{n-1} \) (which may sometimes be convenient) I have arranged a formula and given under it an example in page 9.
Now suppose we put
\[ u_0 + ax + bx^2 + cx^3 + dx^4 + ex^5 + f x^6 = u_0 + ax + \beta x^2 + \gamma x^3 + \delta x^4 \]
rigidly only when \( x = 0 \pm 1 \) and \( \pm 2 \), but within a certain degree of approximation only when \( x \) has any other values; we get directly
\[
\beta = \frac{1}{2} \Delta^3 u - \frac{1}{3} \Delta^4 u
\]
\[
\alpha = \Delta^1 u + \frac{1}{2} \Delta^2 u - \frac{1}{3} \Delta^3 u
\]
\[
\gamma = \frac{1}{2} \Delta^3 u + \frac{1}{3} \Delta^4 u
\]
Whence
\[
\begin{align*}
\Delta u - u &= \frac{1}{2} \Delta^2 u + \frac{1}{2} \Delta^3 u + \frac{1}{4} \Delta^4 u - \frac{1}{6} \Delta^5 u + \frac{5}{12} \Delta^6 u - \frac{1}{2} \Delta^7 u
\end{align*}
\]
\[
+ \frac{1}{2} \Delta^2 u \cdot x + \frac{1}{2} \Delta^3 u \cdot x^2 + \frac{1}{4} \Delta^4 u \cdot x^3 + \frac{5}{12} \Delta^5 u \cdot x^4 - \frac{1}{2} \Delta^7 u \cdot x^5
\]
\[
\text{or} \quad \Delta u - u = \{ x^0 - 3 x^1 - 5 x^2 - 15 x^3 + 4 x^4 + 12 x^5 \} + \frac{1}{2} \Delta^7 u \cdot x^5
\]

Now if we attend to the spirit of Newton's lemma, we see that the errors arising from the use of four differences, when we ought to use six differences, must be nearly maximum when \( x = \pm 1 \pm \frac{1}{2} \), and putting in our equation these values of \( x \), we have rigidly
\[
\begin{align*}
\Delta u - u &= \frac{1}{2} \Delta^2 u + \frac{1}{2} \Delta^3 u + \frac{1}{4} \Delta^4 u - \frac{1}{6} \Delta^5 u + \frac{5}{12} \Delta^6 u - \frac{1}{2} \Delta^7 u
\end{align*}
\]
\[
\text{or} \quad \Delta u - u = \{ x^0 - 3 x^1 - 5 x^2 - 15 x^3 + 4 x^4 + 12 x^5 \} + \frac{1}{2} \Delta^7 u \cdot x^5
\]

Or, in practice, the maximum half error may be taken as
\[
\begin{align*}
.001 (6 \Delta^2 u + 3 \Delta^3 u) \text{ between } u \text{ and } u
\end{align*}
\]
\[
.001 (-6 \Delta^2 u - 2 \Delta^3 u) \text{ between } u \text{ and } u
\]
\[
.001 (-14 \Delta^3 u - 10 \Delta^4 u) \text{ between } u \text{ and } u
\]
\[
.001 (14 \Delta^3 u + 3 \Delta^4 u) \text{ between } u \text{ and } u
\]
and these errors will of course be kept out of our tables; that is, we shall use one place of figures less than that which would be affected.

As an easy (and certainly a very favourable) example of the power of the machine, let us with it calculate a table of the Logarithms of the natural numbers up to 10,000, to five places of decimals. We must first actually calculate by any of the known methods the logarithms of 2, 3, 7, 11, 17, and 19, to seven places of decimals, from which we shall immediately obtain the logarithms of
\[
\begin{align*}
10 & 20 & 40 & 80
11 & 22 & 44 & 84
12 & 24 & 48 & 88
13 & 26 & 52 & 92
14 & 28 & 56 & 96
15 & 30 & 60 & 100
16 & 32 & 64 & 104
17 & 34 & 68 & 108
18 & 36 & 72 & 112
19 & 38 & 76 & 116
\end{align*}
\]

To which, applying the formulae just given, we, by ten easy calculations, and by ten forward and ten backward settings of the machine, shall obtain a stereomoulded table up to 10,000.

The time occupied by the machine at its ordinary rate of working (namely, 120 numbers per hour), would be seventy-five hours; the ten plus and minus (say twenty) settings would not take up quite two hours; the calculations by the formula given of the proper differences to be set in the machine, could not cause a delay of two hours, the time taken (exclusive of the preliminary calculation of the logarithms of 2, 3, 7, 11, 17, and 19) being altogether about seventy-nine, or say eighty hours.

* See next page.
\[
\begin{align*}
\triangle u + \frac{1}{2} \triangle^2 u & = a \\
ax - bx^2 + cx^3 + dx^4 - dx^4 = \triangle u
\end{align*}
\] 

On ruled paper the calculation would appear as below:

\[
\begin{align*}
1\text{st Diff. } x = 10 & \div 2 \\
2\text{nd Diff. } x = 10 & \div 4 \\
3\text{rd Diff. } x = 10 & \div 8 \\
4\text{th Diff. } x = 10 & \div (4 \times 4)
\end{align*}
\]

From which we get Logs. 2600, 2601...3400 or 800 Logarithms.

The calculation for the 1st Diff. is extended five places of decimals further than is necessary, in order to show how \( dx^2 \) and \( dx^4 \) would come in if they were required, and for a similar reason the calculations for the 2nd and 3rd Diffs. are extended respectively four and three places. The semicolon here marks off the 7th place of decimals, and the commas the 15th or lastfigure in the machine.

\[
\begin{align*}
1\text{st Diff. } x & = 10 & 2\text{nd Diff. } x & = 10 & 3\text{rd Diff. } x & = 10 & 4\text{th Diff. } x & = 10 \\
\triangle u & = 10 & \triangle^2 u & = 10 & \triangle^3 u & = 10 & \triangle^4 u & = 10
\end{align*}
\]

As an example let \( u = 352 \) \( 3.71 \), the Sun's Longitude March 13th, 1858, to tabulate the Longitude for March 13th, 14th, 15th...25th.

\[
\begin{align*}
u & = 352 \ 38 \ 3.71
\end{align*}
\]

Putting \( N = 6 \), we have as below:

\[
\begin{align*}
u & = 352 \ 38 \ 3.71
\end{align*}
\]

Giving Sun's Longitude, March 13th to 25th, 1858.
LOGARITHMS

OF

NUMBERS

FROM

1 TO 10,000,

CALCULATED, STEREOMOULDED, AND PRINTED

BY

MACHINERY.